Math 131 Second Midterm Solutions

Problem 1. In each case, find the derivative $y' = \dfrac{dy}{dx}$:

- $y = 2\,x^3 + \dfrac{1}{x^2} - \pi\,x$

$$y = 2x^3 + x^{-2} - \pi x \Longrightarrow \frac{dy}{dx} = 6x^2 - 2x^{-3} - \pi = 6x^2 - \frac{2}{x^3} - \pi.$$

- $y = \left(x + \dfrac{1}{x}\right)^{10}$

  Set $u = x + \dfrac{1}{x}$ so $y = u^{10}$. By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 10u^9 \cdot \left(1 - \frac{1}{x^2}\right) \\ &= 10\left(x + \frac{1}{x}\right)^9 \left(1 - \frac{1}{x^2}\right).\end{aligned}$$

- $y = \sqrt{3 + 2x^2}$

  Set $u = 3 + 2x^2$ so $y = \sqrt{u}$. By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}} \cdot (4x) \\ &= \frac{2x}{\sqrt{3 + 2x^2}}.\end{aligned}$$

Problem 2. Find the equation of the tangent line to the graph of the function y defined implicitly by

$$(x + y)^2 = y$$

at the point $(0, 1)$.

We first find $y' = \dfrac{dy}{dx}$ by implicit differentiation:

$$\begin{aligned}(x+y)^2 = y &\Longrightarrow 2(x+y)\cdot(x+y)' = y' \\ &\Longrightarrow 2(x+y)\cdot(1+y') = y' \\ &\Longrightarrow 2x + 2xy' + 2y + 2yy' = y' \\ &\Longrightarrow 2xy' + 2yy' - y' = -2x - 2y \\ &\Longrightarrow y'(2x+2y-1) = -2x-2y \\ &\Longrightarrow y' = \frac{-2x-2y}{2x+2y-1}.\end{aligned}$$

The slope of the tangent line is the derivative y' at the point $(0,1)$, so

$$y' \text{ at } (0,1) = \frac{-2(0)-2(1)}{2(0)+2(1)-1} = -2.$$

Hence the equation of the tangent line is

$$y - 1 = -2(x-0) \qquad \text{or} \qquad y = -2x+1.$$

Problem 3. When the price of a certain commodity is p dollars per unit, the consumers buy x hundred units of the commodity, where

$$2\,p^2 + 98\,p\,x = 78400.$$

Here, both p and x are functions of time measured in months. How fast is x changing with respect to time when the unit price is \$70 and is decreasing at the rate of \$2 per month?

We want to find $x' = \dfrac{dx}{dt}$ when $p = 70$ and $p' = \dfrac{dp}{dt} = -2$. First find x when $p = 70$:

$$\begin{aligned}2\,p^2 + 98\,p\,x = 78400 &\Longrightarrow 2(70)^2 + (98)(70)x = 78400 \\ &\Longrightarrow x = \frac{78400-9800}{6860} = 10.\end{aligned}$$

Now differentiate the given relation between p and x with respect to time and substitute for what is known:

$$\begin{aligned}2\,p^2 + 98\,p\,x = 78400 &\Longrightarrow 4\,p\,p' + 98(p'\,x + p\,x') = 0 \\ &\Longrightarrow 4(70)(-2) + 98((-2)(10) + 70x') = 0 \\ &\Longrightarrow -560 - 1960 + 6860x' = 0 \\ &\Longrightarrow x' = \frac{2520}{6860} = 0.36\end{aligned}$$

Since x is the demand in hundred units, it follows that the demand increases at the rate of 36 units per month.

Problem 4. Sketch a possible graph for a function f which has all the following properties; determine the horizontal and vertical asymptotes, the critical points and their type, as well as the inflection points:

- The domain of f is all real x except $x = -1$
- $\lim_{x\to -1} f(x) = +\infty$ and $\lim_{x\to \pm\infty} f(x) = 2$
- $f'(x)$ is $\begin{cases} \text{positive when } x < -1 \text{ and } 0 < x < 3 \\ \text{negative when } -1 < x < 0 \text{ and } x > 3 \end{cases}$
- $f''(x)$ is $\begin{cases} \text{positive when } x < 1 \text{ and } x > 4 \\ \text{negative when } 1 < x < 4 \end{cases}$

It is clear from the second condition that the line $x = -1$ is the vertical asymptote and the line $y = 2$ is the horizontal asymptote of f. We put the remaining information in the following table:

x	$-\infty$		-1		0		1		3		4		$+\infty$
$f''(x)$		$+++$	$\Vert$	$+++$	$\vert$	$+++$	$\vert$	$---$	$\vert$	$---$	$\vert$	$+++$	
$f'(x)$		$+++$	$\Vert$	$---$	$\vert$	$+++$	$\vert$	$+++$	$\vert$	$---$	$\vert$	$---$	
$f(x)$		$\nearrow$ $\smile$	$\Vert$	$\searrow$ $\smile$	$\vert$	$\nearrow$ $\smile$	$\vert$	$\nearrow$ $\frown$	$\vert$	$\searrow$ $\frown$	$\vert$	$\searrow$ $\smile$	

The critical points are $x = -1$ (neither), $x = 0$ (local min) and $x = 3$ (local max). The inflection points are $x = 1$ and $x = 4$. Here is a possible sketch: