Math 143 First Midterm Solutions

Problem 1. [8 points] Find the area under the curve $y = x\,e^{-x}$ between $x = 0$ and $x = 2$.

The required area is the integral of $x\,e^{-x}$ from $x = 0$ to $x = 2$. We use integration by parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

to compute this integral. Set

$$\begin{cases} u & = x \\ dv & = e^{-x} \end{cases} \qquad \text{so} \qquad \begin{cases} du & = dx \\ v & = -e^{-x}. \end{cases}$$

Then,

$$\begin{aligned} \text{area} &= \int_0^2 x\,e^{-x}\,dx = -x\,e^{-x}\Big|_0^2 + \int_0^2 e^{-x}\,dx \\ &= -2e^{-2} - e^{-x}\Big|_0^2 \\ &= -2e^{-2} - e^{-2} + 1 \\ &= 1 - 3e^{-2} \approx 0.593994 \end{aligned}$$

Problem 2. [18 points] Integrate:

(i) $\displaystyle\int x^2\sqrt{1+x^3}\,dx$

This is a standard substitution. Set $u = 1 + x^3$, so $du = 3x^2\,dx$. Then,

$$\begin{aligned} \int x^2\sqrt{1+x^3}\,dx &= \int \sqrt{u}\cdot\tfrac{1}{3}\,du = \tfrac{1}{3}\int u^{1/2}\,du \\ &= \tfrac{2}{9}\,u^{3/2} + C = \tfrac{2}{9}\,(1+x^3)^{3/2} + C. \end{aligned}$$

(ii) $\displaystyle\int \sqrt{4-x^2}\,dx$

We use a trigonometric substitution. Set $x = 2\sin\theta$, so $dx = 2\cos\theta\,d\theta$.

Then,

$$\begin{aligned}\int \sqrt{4-x^2}\,dx &= \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta \\ &= 4\int \sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta \\ &= 4\int \cos^2\theta\,d\theta \\ &= 2\int (1+\cos(2\theta))\,d\theta \qquad \text{(doubling angle formula)} \\ &= 2\theta + \sin(2\theta) + C.\end{aligned}$$

To write this answer in terms of x, note that

$$\theta = \sin^{-1}\left(\tfrac{x}{2}\right)$$

and

$$\sin(2\theta) = 2\sin\theta\cdot\cos\theta = x\cdot\sqrt{1-\left(\tfrac{x}{2}\right)^2} = \tfrac{1}{2}x\sqrt{4-x^2}.$$

It follows that

$$\int \sqrt{4-x^2}\,dx = 2\sin^{-1}\left(\tfrac{x}{2}\right) + \tfrac{1}{2}x\sqrt{4-x^2} + C.$$

(iii) $\displaystyle\int \frac{1}{x(x^2+4)}\,dx$

We use partial fractions. Write

$$\frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

To determine the constants A, B, C, find the numerator of the combined fraction on the right and set it equal to the numerator on the left:

$$1 = A(x^2+4) + (Bx+C)x = (A+B)x^2 + Cx + 4A.$$

Since this is to hold for every x, it follows that

$$A+B=0, \qquad C=0, \qquad 4A=1$$

or

$$A=\tfrac{1}{4}, \qquad B=-\tfrac{1}{4}, \qquad C=0.$$

Thus,

$$\begin{aligned}\int \frac{1}{x(x^2+4)}\,dx &= \tfrac{1}{4}\int \frac{1}{x}\,dx - \tfrac{1}{4}\int \frac{x}{x^2+4}\,dx \\ &= \tfrac{1}{4}\ln|x| - \tfrac{1}{4}\cdot\tfrac{1}{2}\ln|x^2+4| + C \quad \text{(substitution } u = x^2+4\text{)} \\ &= \tfrac{1}{4}\ln|x| - \tfrac{1}{8}\ln(x^2+4) + C.\end{aligned}$$

Problem 3. [8 points] Find $\lim_{x\to 0}\left(1-5x^2\right)^{1/x^2}$ (L'Hôpefully you remember what to do).

As $x \to 0$, the base $1-5x^2$ tends to 1 and the exponent $1/x^2$ tends to ∞, so this limit has the indeterminate form 1^∞. Call

$$L = \lim_{x\to 0}\left(1-5x^2\right)^{1/x^2}.$$

Then,

$$\begin{aligned}
\ln L &= \lim_{x\to 0} \ln\left[\left(1-5x^2\right)^{1/x^2}\right] \\
&= \lim_{x\to 0} \frac{\ln(1-5x^2)}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{-10x}{1-5x^2}}{2x} && \text{(L'Hôpital's rule)} \\
&= \lim_{x\to 0} \frac{-5}{1-5x^2} && \text{(cancel } 2x\text{)} \\
&= -5.
\end{aligned}$$

It follows that $L = e^{-5}$.

Problem 4. [6 points] Use your calculator to find the Midpoint and Simpson approximations to the integral

$$I = \int_0^1 \sin(x^4)\, dx$$

with $n = 10$, 20 and 50 subdivisions (round your answers to six decimal places). Based on your results, estimate the value of I to four decimal places.

We use the TI program ISUMS for $Y_1 = \sin(X^4)$, $A = 0, B = 1$ and $N = 10, 20, 50$. Here are the results:

	$n = 10$	$n = 20$	$n = 50$
M	0.186652	0.187343	0.187533
S	0.187565	0.187569	0.187569

This experiment suggests a value of $I = 0.1875$ up to four decimal places.