Math 143 Second Midterm Solutions

Problem 1. [9 points] Let $p > 0$ be a fixed number. Show that the improper integral $\int_0^\infty x\, e^{-px^2}\, dx$ converges and find its value.

The substitution

$$u = -px^2 \qquad du = -2px\, dx$$

shows that

$$\int x\, e^{-px^2}\, dx = -\frac{1}{2p} \int e^u\, du = -\frac{1}{2p} e^u + C = -\frac{1}{2p} e^{-px^2} + C.$$

Hence,

$$\begin{aligned} \int_0^\infty x\, e^{-px^2}\, dx &= \lim_{b\to\infty} \int_0^b x\, e^{-px^2}\, dx \\ &= \lim_{b\to\infty} -\frac{1}{2p} e^{-px^2} \Big|_0^b \\ &= \lim_{b\to\infty} -\frac{1}{2p} (e^{-pb^2} - e^0). \end{aligned}$$

But $\lim_{b\to\infty} e^{-pb^2} = 0$ since $p > 0$ and $e^0 = 1$. It follows that our improper integral converges and

$$\int_0^\infty x\, e^{-px^2}\, dx = \frac{1}{2p}.$$

Problem 2. [9 points] In each case, find $\lim\limits_{n\to\infty} a_n$:

(i) $a_n = \cos(n\pi)$

We know from the definiton of the cosine function that $\cos(n\pi)$ is 1 if n is even and is -1 if n is odd. In other words, $\cos(n\pi) = (-1)^n$ which takes the values 1 and -1 in an alternating pattern. Clearly this sequence diverges, so $\lim_{n\to\infty} a_n$ does not exist.

(ii) $a_n = \dfrac{\sqrt{n}}{n+4}$

Here the numerator and denominator both grow arbitrarily large as $n \to \infty$. To see what happens to their ratio, it helps if we first divide both by n:

$$a_n = \frac{\sqrt{n}}{n+4} = \frac{\dfrac{\sqrt{n}}{n}}{\dfrac{n+4}{n}} = \frac{\dfrac{1}{\sqrt{n}}}{1 + \dfrac{4}{n}}.$$

It is now clear that

$$\lim_{n\to\infty} a_n = \frac{0}{1+0} = 0.$$

(iii) $\{a_n\}$ is a sequence such that $\cos\left(\frac{1}{n}\right) \le a_n \le 1 + \frac{2017}{n^2}$ for all n.

As $n \to \infty$, we have $1/n \to 0$, so $\cos(1/n) \to \cos(0) = 1$. Also $1/n^2 \to 0$, so $1 + 2017/n^2 \to 1$. The sandwich lemma now shows that $\lim_{n\to\infty} a_n = 1$.

Problem 3. [12 points] Determine the convergence or divergence of the following series. In each case, specify the test that you are using:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n^2-1}$

For large n the general term $\dfrac{\sqrt{n}}{3n^2-1}$ behaves like $\dfrac{\sqrt{n}}{3n^2} = \dfrac{1}{3n^{3/2}}$. This suggests that we compare our series with $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$:

$$\lim_{n\to\infty} \frac{\dfrac{\sqrt{n}}{3n^2-1}}{\dfrac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^2}{3n^2-1} = \lim_{n\to\infty} \frac{1}{3-\dfrac{1}{n^2}} = \frac{1}{3} > 0.$$

Since the series $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges (p-series with $p = 3/2 > 1$), the limit comparison test shows that our series must converge too.

(ii) $\displaystyle\sum_{n=1}^{\infty} n\,e^{-n^2}$

Consider the function $x\,e^{-x^2}$ which is positive and decreasing for $x > 1$. By a computation similar to problem 1 with $p = 1$, the improper integral

$$\begin{aligned}\int_1^{\infty} x\,e^{-x^2}\,dx &= \lim_{b\to\infty} \int_1^{b} x\,e^{-x^2}\,dx \\ &= \lim_{b\to\infty} -\frac{1}{2}\,e^{-x^2}\Big|_1^{b} \\ &= \lim_{b\to\infty} -\frac{1}{2}(e^{-b^2} - e^{-1}) = \frac{1}{2e}\end{aligned}$$

converges. Hence, by the integral test, the series $\sum_{n=1}^{\infty} n\,e^{-n^2}$ converges too.

(iii) $\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)(n+1)}$

We note that

$$\lim_{n\to\infty} \frac{n^2}{(2n-1)(n+1)} = \lim_{n\to\infty} \frac{n^2}{2n^2+n-1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}-\frac{1}{n^2}} = \frac{1}{2} \neq 0.$$

Hence, by the basic divergence test, the series diverges.

Problem 4. [10 points] Find the value(s) of x such that

$$\sum_{n=2}^{\infty} (1+x)^{-n} = \frac{1}{6}.$$

Recall the geometric series formula

$$\sum_{n=2}^{\infty} r^n = r^2 + r^3 + r^4 + \cdots = \frac{r^2}{1-r}$$

which is valid for $|r| < 1$. Applying this formula for $r = 1/(1+x)$, we see that

$$\sum_{n=2}^{\infty} (1+x)^{-n} = \sum_{n=2}^{\infty} \left(\frac{1}{1+x}\right)^n = \frac{\frac{1}{(1+x)^2}}{1-\frac{1}{1+x}} = \frac{\frac{1}{(1+x)^2}}{\frac{x}{1+x}} = \frac{1}{x^2+x} = \frac{1}{6}.$$

This gives $x^2 + x = 6$ or $x^2 + x - 6 = 0$, which by the quadratic formula has two solutions $x = 2$ and $x = -3$. It is easy to check that both solutions indeed work. For $x = 2$, we obtain the series

$$\sum_{n=2}^{\infty} 3^{-n} = \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{1}{6},$$

while for $x = -3$ we obtain the series

$$\sum_{n=2}^{\infty} (-2)^{-n} = \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{\frac{1}{4}}{1+\frac{1}{2}} = \frac{1}{6}.$$