MAT 160, PROBLEM SEMINAR, WEEK OF 2/22/99

PROBLEM SET 5: MORE ON COUNTING

For the following problems, you need to know these facts:
• The number of permutations of n different objects is

$$n! = n(n-1)\cdots 2\cdot 1$$

with the convention $0! = 1$.
Example: There are $10! = 3628800$ different ways to seat 10 students in a row.
• If we have n_1 identical objects of type 1, n_2 identical objects of type 2, ... and finally n_k identical objects of type k, the number of permutations of all these $n_1 + n_2 + \cdots + n_k$ objects is

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!\, n_2! \cdots n_k!}$$

Example: Having 2 red, 3 green and 5 blue balls, there are $(2+3+5)!/(2!3!5!) = 2520$ different ways to arrange them in a row.
• The number of different ways to choose k objects from a set of n objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Here $0 \le k \le n$. $\binom{n}{k}$ is pronounced "n choose k."
Example: There are $\binom{10}{3} = 10!/(3!7!) = 120$ different ways to form a team of 3 students out of a group of 10.
• The numbers $\binom{n}{k}$ form the so-called *binomial coefficients*. The *Binomial Theorem* states that

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n.$$

for any a, b. For $n = 2$ this reduces to the familiar formula $(a+b)^2 = a^2 + 2ab + b^2$, and for $n = 3$ to $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Problem 29. In Deep Blue Chess Club there are 2 girls and 7 boys. A team of 4 players is to be chosen for a tournament, and there must be at least a girl on the team. In how many different ways can they choose their team?

Problem 30. Count the number of ways one can group 48 distinct people into 24 pairs. Can you find the answer in the general case of $2n$ people to form n pairs?

Problem 31. How many different necklaces can you make using 5 distinct beads? Arrangements can be rotated or flipped; two necklaces are considered the same if one can be obtained from the other by rotating and/or flipping.

Problem 32. Prove the following identities:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Problem 33. (a) Using the Binomial Theorem, prove the following two identities:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0$$

(b) Using the first identity above, try to find another proof for the fact that a set of n-elements has exactly 2^n subsets. (*Hint*: Count the number of k-element subsets of an n-element set for every k between 0 and n.)

Problem 34. We know that $1 + 2 + \cdots + n = n(n+1)/2 = \binom{n+1}{2}$. Try to find a "counting argument" to prove this result.

Problem 35. Below you see the map of a town. All the streets are one-way, so that you can drive only "east" (E) or "north" (N). In how many different ways can you reach point B starting from point A? (*Hint*: Every possible path corresponds to a word having eight E's and five N's, and every such word determines a path. For example, the path shown in bold corresponds to the word ENNEEENEENNEE.)