MAT 160, PROBLEM SEMINAR, WEEK OF 4/5/99

PROBLEM SET 8: RECURSION

Here are few definitions and facts you need for this week's problems.

• Consider a sequence of real numbers $a_1, a_2, \ldots, a_n, \ldots$. Very informally, we say that this sequence is *recursively defined* if each member a_n can be calculated only knowing the index n and the previous members $a_1, \ldots, a_{n-1}$. For simplicity, we often refer to a recursively defined sequence as a *recursion*.

• Example 1. $a_1 = 2$ and for all $n \geq 2$, $a_n = 2a_{n-1}$. It is easy to check that $a_2 = 4, a_3 = 8, a_4 = 16, a_5 = 32$ and in general $a_n = 2^n$.

• Example 2. $a_1 = 1$ and for all $n \geq 2$, $a_n = a_{n-1} + n$. We get $a_2 = a_1 + 2 = 1 + 2$, $a_3 = a_2 + 3 = 1 + 2 + 3$, $a_4 = a_3 + 4 = 1 + 2 + 3 + 4$, and in general, it is easy to check that $a_n = 1 + 2 + \cdots + n = n(n+1)/2$.

• Example 3. $a_1 = a_2 = 1$ and for all $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$ (*Fibonacci sequence*). The first few members of this sequence are

$$a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13, a_8 = 21, \ldots$$

• A recursion is called *first-order* if each a_n can be calculated only knowing n and the previous member a_{n-1}. Examples 1 and 2 above are first-order recursions. To determine a first-order recursion completely, we only need to know the value of the first member a_1.

• A recursion is called *second-order* if each a_n can be calculated only knowing n and the two previous members a_{n-1} and a_{n-2}. Fibonacci sequence is an example of a second-order recursion. To determine a second-order recursion completely, we only need to know the values of the first two members a_1 and a_2.

• The solution to the first-order recursion $a_n = Ca_{n-1}$ (C is a constant) is $a_n = A\,C^n$ for all $n \geq 1$, where the constant A can be determined by the initial value a_1. For example, the recursion $a_n = 2a_{n-1}$ of Example 1 has the solution $a_n = A\,2^n$ for all $n \geq 1$. To determine the constant A, we use the initial condition $a_1 = 2 = A\,2^1$ which gives $A = 1$. Hence $a_n = 2^n$.

• To solve the second-order recursion $a_n = Ca_{n-1} + Da_{n-2}$ (C and D are constants), we form the *characteristic equation* $x^2 - Cx - D = 0$. If this equation has two distinct real roots α and β, then the solution has the form $a_n = A\alpha^n + B\beta^n$ for some constants A and B which can be determined by the initial conditions a_1 and a_2. On the other hand, if the characteristic equation has a repeated root α, then the solution is of the form $a_n = (A + Bn)\alpha^n$, where again the constants A and B can be determined by a_1 and a_2.

Problem 50. Determine which of the following relations define a recursion. Check if the recursion is first- or second-order.

$$a_n = n^2 + 1 \ (n \geq 1), \ \ a_n = na_{n+1} \ (n \geq 1), \ \ a_n = \frac{a_{n-1} + a_{n+1}}{2} \ (n \geq 2), \ \ a_n = \frac{a_{n+2}}{3} \ (n \geq 1)$$

Problem 51. Suppose that $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 3$ and $a_1 = 1$ and $a_2 = 3$. Find a formula for a_n.

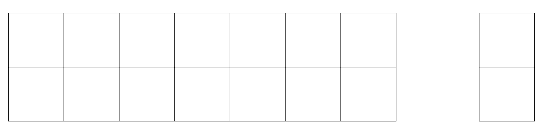
Problem 52. At the start of each year, \$100 is added to a savings account. At the end of each year, interest equal to 5 percent of the amount in the account is added by the bank. Let a_n be the amount of money in the account after the interest payment in the n-th year. Obtain a recursion for a_n.

Problem 53. *A formula for the Fibonacci sequence.* Recall that the Fibonacci sequence is defined by the recursion $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. Form the characteristic equation and solve the recursion to show that

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Problem 54. *Regions in the plane.* A "configuration" of lines is a finite collection of lines in the plane such that each pair of lines has one common point and no three lines have a common point. Let a_n be the number of regions created by a configuration of n lines. Thus, $a_1 = 2$, $a_2 = 4$, etc. By a simple geometric argument, show that $a_n = a_{n-1} + n$ for $n \geq 2$. Conclude that $a_n = 2 + 2 + 3 + 4 + \cdots + n = 1 + n(n+1)/2$.

Problem 55. In how many ways can you tile the 2-by-7 checkerboard on the left using 7 identical dominos on the right? (*Hint*: Let a_n be the number of ways to tile a 2-by-n checkerboard using n identical dominos. Clearly, $a_1 = 1$, $a_2 = 2$, etc. Obtain a recursion for a_n to determine a_7.)



Problem 56. *Tower of Hanoi.* This is a puzzle with three pegs and seven rings of different sizes that could slide onto the pegs. Starting with all rings on one peg in order by size as in the picture, the problem is to transfer the pile to another peg subject to two conditions: rings are moved one at a time, and no ring can ever be placed on top of a smaller ring. How many moves are required to do this? (*Hint*: Consider the general case: Let a_n denote the number of moves required to transfer a pile of n rings to another peg. Thus, $a_1 = 1$, $a_2 = 3$, etc. Show that the recursion $a_n = 2a_{n-1} + 1$ holds for all $n \geq 2$. Conclude that $a_n = 2^n - 1$. In particular, $a_7 = 2^7 - 1 = 127$. Legend has it that an order of monks had a similar puzzle with 64 large golden disks. They supposedly believed that the world would crumble when the job was finished. Our formula shows that $a_{64} = 2^{64} - 1$. If each move only takes 1 second, the job would take more than 10^{11} years! You can find more on the history of this problem and its variations online by searching "tower of hanoi" using any search engine. You can even play the game at http://www.mazeworks.com/hanoi/hanoi.htm)