Here is a rigorous proof of Fubini's Theorem on the equality of double and iterated integrals. The present version is slightly more general than the one stated in the textbook.

Fubini's Theorem. *Let f be an integrable function on the rectangle $R = [a, b] \times [c, d]$. Suppose that for each $y \in [c, d]$, the integral $\int_a^b f(x, y)\,dx$ exists and moreover $\int_a^b f(x, y)\,dx$ as a function of y is integrable on $[c, d]$. Then*

$$\iint_R f\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy.$$

Of course a similar statement holds with the role of x, y changed. When f is continuous on the rectangle R, all the integrability assumptions hold automatically and we have

$$\iint_R f\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx.$$

The proof begins as follows. Take arbitrary partitions $x_0 = a < x_1 < \cdots < x_m = b$ of $[a, b]$ and $y_0 = c < y_1 < \cdots < y_n = d$ of $[c, d]$. Let $\mathcal{P}$ be the partition of R into the mn sub-rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. As usual, set

$$\Delta x_i = x_i - x_{i-1} \qquad \Delta y_j = y_j - y_{j-1} \qquad m_{ij} = \inf_{R_{ij}} f \qquad M_{ij} = \sup_{R_{ij}} f.$$

Since

$$m_{ij} \le f(x, y) \le M_{ij} \qquad \text{for } (x, y) \in R_{ij},$$

the comparison property of the integral in dimension 1 shows that

$$m_{ij}\,\Delta x_i \le \int_{x_{i-1}}^{x_i} f(x, y)\,dx \le M_{ij}\,\Delta x_i \qquad \text{if } y \in [y_{j-1}, y_j].$$

Summing over all i from 1 to m, we obtain

$$\sum_{i=1}^m m_{ij}\,\Delta x_i \le \int_a^b f(x, y)\,dx \le \sum_{i=1}^m M_{ij}\,\Delta x_i \qquad \text{if } y \in [y_{j-1}, y_j].$$

Applying comparison once more gives

$$\left(\sum_{i=1}^m m_{ij}\,\Delta x_i\right)\Delta y_j \le \int_{y_{j-1}}^{y_j} \int_a^b f(x, y)\,dx\,dy \le \left(\sum_{i=1}^m M_{ij}\,\Delta x_i\right)\Delta y_j.$$

Summing up over j from 1 to n, we obtain

$$\sum_{j=1}^n \sum_{i=1}^m m_{ij}\,\Delta x_i\,\Delta y_j \le \int_c^d \int_a^b f(x, y)\,dx\,dy \le \sum_{j=1}^n \sum_{i=1}^m M_{ij}\,\Delta x_i\,\Delta y_j.$$

Thus, for every partition $\mathcal{P}$ of the rectangle R,

$$L(f, \mathcal{P}) \leq \int_c^d \int_a^b f(x, y)\, dx\, dy \leq U(f, \mathcal{P}).$$

On the other hand, since by the assumption f is integrable on R, the double integral $\iint_R f\, dA$ is the *unique* number which satisfies

$$L(f, \mathcal{P}) \leq \iint_R f\, dA \leq U(f, \mathcal{P})$$

for every partition $\mathcal{P}$. Therefore, we must have

$$\iint_R f\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy,$$

as claimed.