Math 328 Midterm Exam
March 20, 2007

- The exam has 5 problems, each worth 20 points.
- Return your work on Thursday 5/22 at the start of lecture.
- Solutions must be complete and clear, showing all the steps along the way. Neatness counts!
- You can use any result that has been discussed in lecture, without re-doing, but make sure you clearly quote the theorems that you invoke.
- Any form of communication with other people regarding this test is absolutely forbidden. You may consult any book on the subject, but give proper credit if you end up using sources other than the official text.
- If necessary, computations such as complicated integrals can be performed with the aid of a table, calculator or computer.

Problem 1. Consider the 2-periodic function defined by

$$f(x) = \begin{cases} 1 & -1 < x \leq 0 \\ x & 0 < x \leq 1 \end{cases} \qquad \text{and} \qquad f(x+2) = f(x).$$

(i) Sketch the graph of f and its Fourier series $FS(f)$ over $[-3, 3]$. At what points x on the real line is it true that $FS(f)(x) = f(x)$?

(ii) Compute $FS(f)$.

(iii) Use the result of part (ii) to evaluate the sum

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Problem 2. The temperature $u(x,t)$ of a metal rod of unit length with insulated ends satisfies

$$\begin{cases} u_t = u_{xx} & 0 < x < 1,\ t > 0 \\ u_x(0,t) = u_x(1,t) = 0 & t > 0 \\ u(x,0) = x^2 & 0 < x < 1 \end{cases}$$

(i) Find the (formal) solution of this equation. Identify the steady-state and transient temperatures.

(ii) Use (i) to estimate the temperature at the midpoint of the rod at $t = 0.05$.

Problem 3. Consider the boundary-initial value problem

$$\begin{cases} u_t = u_{xx} + \dfrac{1}{9}u & 0 < x < \pi,\ t > 0 \\ u(0,t) = A,\ u(\pi,t) = B & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \end{cases}$$

where A, B are real constants.

(i) Find a time-independent solution $v(x)$ of the above PDE which satisfies the non-homogeneous boundary conditions as well.

(ii) Set up the corresponding problem for $w(x,t) = u(x,t) - v(x)$ which now has homogeneous boundary conditions. Then use the method of separation of variables to solve it.

(iii) Find the (formal) solution $u(x,t)$ of the original problem by adding $v(x)$ that you found in (i) to $w(x,t)$ that you found in (ii).

Problem 4. The vibrations of a finite string satisfies the wave equation

$$\begin{cases} u_{tt} = c^2\, u_{xx} & 0 < x < 6,\ t > 0 \\ u(0,t) = u(6,t) = 0 & t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = 0 & 0 < x < 6 \end{cases}$$

where

$$f(x) = \begin{cases} \frac{1}{2}\, x & 0 < x \leq 4 \\ -x + 6 & 4 < x < 6 \end{cases}$$

Carefully sketch the profile of the string at times $t = \frac{1}{c}$, $t = \frac{2}{c}$ and $t = \frac{3}{c}$.

Problem 5. Consider a semi-infinite string that is positioned along the interval $0 \leq x < \infty$. Assume that the string is initially at rest, but its left end at $x = 0$ moves up or down according to a given function of time $h(t)$. Thus, the displacement $u(x,t)$ of the string satisfies the following boundary-initial value problem:

$$\begin{cases} u_{tt} = c^2\, u_{xx} & 0 < x < \infty,\ t > 0 \\ u(0,t) = h(t) & t > 0 \\ u(x,0) = u_t(x,0) = 0 & 0 < x < \infty. \end{cases}$$

(i) Find the solution $u(x,t)$ of the above problem. To this end, take the general solution

$$u(x,t) = \phi(x - ct) + \psi(x + ct) \qquad 0 < x < \infty,\ t > 0$$

and apply the initial conditions and then the boundary condition to find ϕ and ψ.

(ii) Using your answer to (i), verify that for each given $t > 0$, the part of the string to the right of the point $x = ct$ remains undisturbed prior to time t.

(iii) Apply the result of part (i) to find $u(x,t)$ in the case

$$h(t) = \begin{cases} \sin(\pi t) & 0 < t \leq 1 \\ 0 & t \geq 1 \end{cases}$$

(Intuitively, the left end is lifted up 1 unit and then returned to the origin where it remains after time $t = 1$.) Sketch the profile of the string at a time $t > 1$.