MATH 360 Final Exam

December 15, 2001

- Total points: 80
- Solutions must be written completely, carefully and clearly. You must justify all your claims. If you are referring to a result that has already been proved in class, state that result clearly.
- You are not allowed to share your ideas or solutions with others, but you may use your notes or any book on the subject.
- The exam is due 10:00 am of Monday 12/17. Hand in your work to me at DRL 4N59, or simply slip it under the door in case I am not there. Solutions will not be accepted after the due time.

Problem 1. [4 points] Suppose x and y are real numbers such that $|x - y| < |x|$. Show that $xy > 0$.

Problem 2. [5 points] Let $\{x_n\}_{n\geq 1}$ be any sequence in $[0, 1]$. Define a new sequence $\{y_n\}_{n\geq 1}$ by

$$y_n = \max_{1\leq i\leq n} x_i.$$

Prove that $\{y_n\}_{n\geq 1}$ converges.

Problem 3. [5 points] Let X and Y be two sets, $f : X \to Y$ be a map and $A \subset Y$. Show that $f(f^{-1}(A)) \subset A$. Show that if f is onto, then $f(f^{-1}(A)) = A$.

Problem 4. [8 points] Let A and B be non-empty sets in a metric space. Show that

$$\overline{A \cap B} \subset \overline{A} \cap \overline{B}.$$

Show by an example that the above inclusion can be proper.

Problem 5. [9 points] Provide an example for

(i) A continuous map $f : X \to Y$ and an open set $A \subset X$ such that the image $f(A) \subset Y$ is not open.

(ii) A continuous map $f : X \to Y$ and a disconnected set $A \subset X$ such that the image $f(A) \subset Y$ is connected.

(iii) A continuous map $f : X \to Y$ and a compact set $A \subset Y$ such that the preimage $f^{-1}(A) \subset X$ is not compact.

Problem 6. [8 points] Let $\{x_n\}_{n\geq 1}$ be a sequence of non-zero vectors in $\mathbb{R}^n$ such that

$$\lim_{n\to\infty} \frac{\|x_{n+1}\|}{\|x_n\|} = \frac{1}{2}.$$

Show that $\{x_n\}$ converges to the origin $0 \in \mathbb{R}^n$. (Hint: Choose any $\frac{1}{2} < \lambda < 1$ and observe that $\|x_{n+1}\|/\|x_n\| < \lambda$ for all sufficiently large n.)

Problem 7. [6 points] Show that for every integer n, there exists a point x such that $|x - n\pi| < \dfrac{\pi}{2}$ and $x = \tan x$.

Problem 8. [10 points] Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

Find all points at which f is continuous.

Problem 9. [10 points] Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$, and the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$. Fix two vectors $x, y \in V$. Suppose $\|x - ty\|$ takes its minimum value over all $t \in \mathbb{R}$ exactly when $t = 0$. Show that $\langle x, y \rangle = 0$.

Problem 10. [15 points] Let $f : [0, 1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$.

(i) Show that there exists an x such that $f(x) = f(x + \frac{1}{2})$.

(ii) Let n be any natural number. Show that there exists an x such that $f(x) = f(x + \frac{1}{n})$.

(iii) [Bonus problem] Let $0 < a < 1$ be a real number which is not equal to $\frac{1}{n}$ for any natural number n. Show that there is a continuous function $f : [0, 1] \to \mathbb{R}$ with $f(0) = f(1)$ such that $f(x) \neq f(x + a)$ for all x.

(Hint: For the first part, consider the continuous function $g(x) = f(x + \frac{1}{2}) - f(x)$ and assume it never vanishes to get a contradiction.)