MATH 360 Homework 10

If God is perfect, why did He
create discontinuous functions?

Problem 1. Using the $\varepsilon - \delta$ definition of continuity, prove carefully that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x + y$ is continuous on $\mathbb{R}^2$.

Problem 2. Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a closed set $A \subset \mathbb{R}$ such that $f(A)$ is not closed.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Decide which of the following sets must be open, closed, compact, or connected (in each case, either prove or give a counterexample):

(i) $\{x \in \mathbb{R} : f(x) = 0\}$

(ii) $\{x \in \mathbb{R} : f(x) > 0\}$

(iii) $\{f(x) : x \geq 0\}$

(iv) $\{f(x) : 0 \leq x \leq 1\}$

Problem 4. Let (M, d) be a metric space and fix some $p \in M$. Define a function $f : M \to \mathbb{R}$ by $f(x) = d(x, p)$. Prove that f is continuous on M.

Problem 5. Let $f : M \to S$ and $g : S \to T$ be maps between metric spaces and define the composition $h = g \circ f : M \to T$ by $h(x) = g(f(x))$ for every $x \in M$.

(i) Prove that if f and g are continuous maps, so is h. (Hint: You may use any of the 3 equivalent conditions for continuity, but perhaps the one using sequences is the easiest here).

(ii) Show by an example that if f and h are continuous maps, g does not have to be continuous at *any* point. (Hint: Consider the function $g : \mathbb{R} \to \mathbb{R}$ which takes the value 0 on rationals and 1 otherwise.)

Problem 6. Let $f, g : M \to S$ be two continuous maps between metric spaces. Prove that the set $E = \{x \in M : f(x) = g(x)\}$ is closed.

Problem 7. Assume $f : [a, b] \to \mathbb{R}$ is continuous with $f(a) = f(b) = 0$ but f is not the constant function 0. Let M be the maximum value of f on $[a, b]$. Prove that for every small $\varepsilon > 0$, the equation $f(x) = M - \varepsilon$ has at least two roots in (a, b).

Reading assignment: Read the following parts of Boas's *A primer of real functions* after Leno's monologue and before you go to bed: 32-38 (on connectivity), 45-52 (on compactness), 52-61 (on convergence and completeness), 77-96 (on continuous functions). Have fun!