MATH 360 Homework 2

What makes the Universe so hard to comprehend is that there is nothing to compare it with.

Problem 1. (a) What is the intersection

$$\bigcap_{n=1}^{\infty} (-\infty, \frac{1}{n}] \ ?$$

Prove your claim carefully.

(b) Use the result of (a) and DeMorgan's Laws to find

$$\left(\bigcup_{n=1}^{\infty} (\frac{1}{n}, +\infty)\right)^c = \mathbb{R} \setminus \left(\bigcup_{n=1}^{\infty} (\frac{1}{n}, +\infty)\right).$$

Problem 2. Define a function $f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 2n - 1$. Is f one-to-one (injective)? Is f onto (surjective)? If $E \subset \mathbb{Z}$ is the set of even integers (positive or negative or zero), what is the preimage $f^{-1}(E)$?

Problem 3. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

(f is usually called the *sign function*.) Let A be the interval $(-2, 1)$ in $\mathbb{R}$. Find the image $f(A)$ and the preimage $f^{-1}(A)$.

Problem 4. Recall that $\mathbb{Z} \times \mathbb{Z}$ is the set of all ordered pairs (m, n) where both m and n are integers. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(m, n) = m + n$. Is f one-to-one? Is f onto?

Problem 5. Let $f : S \to T$ be a function and $A_n \subset T$ for all $n = 1, 2, 3, \ldots$. Prove that

$$f^{-1}(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} f^{-1}(A_n).$$

Problem 6. Let S be the set of all polynomials of the form $p(x) = ax^2 + bx + c$ and T be the set of all polynomials of the form $q(x) = rx + s$. Here a, b, c, r, s can be arbitrary real numbers. Let $D : S \to T$ be the "differentiation" mapping defined by $D(p) = p'$. Here p' is the derivative of p with respect to x. Is D one-to-one? Is D onto?

Problem 7. Show that the set $\mathbb{N}$ of natural numbers and the set $\mathbb{N} \times \mathbb{N}$ (which by definition consists of all pairs of natural numbers of the form (m, n)) have the same power. (Hint: To do this, you must find a bijective map $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. It might be hard to find a formula for such a function. Instead, you can describe a way of "labeling" pairs in $\mathbb{N} \times \mathbb{N}$ by natural numbers. You may find it useful to describe this on a picture of $\mathbb{N} \times \mathbb{N}$.)

Reading assignments. Read pages 1-11 of Boas's *A primer of real functions* up to the end of exercise 3.3. Make sure you think about its exercises.