MATH 360 Homework 3

Radioactive cats have 18 half-lives.

Problem 1. For the following sets $S \subset \mathbb{R}$, find all the upper bounds (if any) and the least upper bound $\sup(S)$:

- $S = \{n + (-1)^n : n \in \mathbb{N}\}$
- $S = \{x \in \mathbb{Q} : 1 \leq x^2 < 5\}$
- $S = \{-k : k \in \mathbb{N}\}$
- $S = \{a, b, c\}$, where $a > b > c$.

Problem 2. Let A and B be non-empty subsets of the real line which are bounded from above (so that their supremums exist). Suppose that for every $x \in A$ you can find a $y \in B$ such that $x \leq y$. Show that $\sup(A) \leq \sup(B)$.

Problem 3. Let $S \subset \mathbb{R}$ be non-empty. When should we call S *bounded from below*? Formulate the definition of *a lower bound* for S. Formulate the definition for *the greatest lower bound* of S. The greatest lower bound of S is often denoted by $\inf(S)$ (the "infimum of S"). When should we set $\inf(S) = -\infty$? Show that if $M = \inf(S) \in \mathbb{R}$, then for every $\varepsilon > 0$ we can find some $x \in S$ such that $M \leq x < M + \varepsilon$.

Problem 4. Show that for all real numbers x and y we have

$$\big|\,|x| - |y|\,\big| \leq |x - y|.$$

This says that the distance between $|x|$ and $|y|$ is less than or equal to the distance between x and y. (Hint: You should prove the two inequalities $-|x-y| \leq |x| - |y| \leq |x-y|$. Both follow from the triangle inequality $|a+b| \leq |a| + |b|$ if you choose a and b cleverly.)

Problem 5. Show that the real line has the *Archimedean* property in the following sense: "Given any $x > 0$ there exists a natural number n such that $0 < x < n$." (Admittedly this looks ridiculously simple, because we are so used to our intuition of numbers that we take this property for granted; Amazingly there are ordered fields -noncomplete of course- which do not have this property! See problem 47 page 102 of Marsden-Hoffman for such an example. The Archimedean property follows from completeness axiom for $\mathbb{R}$. Here is a hint for you: If the property fails, there must be a number $x > 0$ such that every natural number n satisfies $n \leq x$. So x will be an upper bound for $\mathbb{N}$. Then, by the completeness axiom, $\mathbb{N}$ must have a supremum. Now draw a contradiction.)

Problem 6. Use mathematical induction to prove that for every natural number $n \geq 8$, we have $n < (1.3)^n$.

Problem 7. Use mathematical induction to prove that the sequence

$$\left\{ x_n = \frac{2^n}{n!} \right\}_{n \geq 1}$$

is monotonically decreasing.