Math 360 Homework 4

I don't have any solution but I certainly admire the problem.

Problem 1. Use mathematical induction to show that for every integer $n \geq 1$, the number $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9.

Problem 2. (a) Give an example of a non-convergent sequence in $\mathbb{R}$ which is bounded.

(b) Give an example of a non-convergent sequence $\{x_n\}$ in $\mathbb{R}$ such that the sequence $\{|x_n|\}$ is convergent.

Problem 3. Use the definition of limit to show that

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{2^n} = 0$

(b) $\displaystyle\lim_{n\to\infty} \frac{3n}{2n+1} = \frac{3}{2}$

Problem 4. Show that if $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} |x_n| = |L|$. (Hint: Use the definition of limit and the inequality $||a| - |b|| \leq |a - b|$.)

Problem 5. (a) Show that for all $n \geq 1$,

$$0 < \frac{3^n}{n!} \leq 65 \cdot \frac{1}{2^n}.$$

(Hint: Check the right inequality by hand for $1 \leq n \leq 5$, and by mathematical induction for $n > 5$.)

(b) Using part (a) and the fact that $\displaystyle\lim_{n\to\infty} \frac{1}{2^n} = 0$ (problem 3(a)), show that $\displaystyle\lim_{n\to\infty} \frac{3^n}{n!}$ exists and find its value.

Problem 6. Define a sequence $\{x_n\}$ in $\mathbb{R}$ by setting $x_1 = \frac{1}{2}$, and $x_{n+1} = \sqrt{x_n}$ for every $n \geq 1$. Show that $\{x_n\}$ converges. (Hint: Argue that this sequence is increasing and bounded above.) Can you guess what $\lim_{n\to\infty} x_n$ is?

Problem 7. Suppose that the sequence $\{x_n\}$ in $\mathbb{R}$ satisfies $|x_{n+1} - x_n| \leq \dfrac{1}{3^n}$ for all $n \geq 1$. Prove that $\{x_n\}$ converges. (Hint: Show that $\{x_n\}$ is a Cauchy sequence.)