# MATH 360 Homework 5

*Friends come and go but*
*enemies accumulate.*

**Problem 1.** (a) Give an example of an unbounded sequence $\{x_n\}$ in $\mathbb{R}$ which has a subsequence converging to 0.

(b) Give an example of a sequence $\{x_n\}$ in $\mathbb{R}$ which has 3 subsequences converging to $-1, 1$, and 5.

**Problem 2.** Show that the sequence $x_n = 2e^{\cos(3n)}$, $n = 1, 2, 3, \ldots$, has a convergent subsequence.

**Problem 3.** Let $\{x_n\}$ be a sequence in $\mathbb{R}$ which converges to $L$. Suppose that $\{y_n\}$ is another sequence in $\mathbb{R}$ with the property that $|x_n - y_n| \leq \dfrac{1}{n^2}$ for all $n$. Prove that $\{y_n\}$ converges to $L$ also.

**Problem 4.** Let $0 < a < 1$ and $\{x_n\}$ be a sequence in $\mathbb{R}$ with the property that

$$|x_n - x_{n-1}| \leq a|x_{n-1} - x_{n-2}|$$

for all $n \geq 3$.

(a) Show that for all $n \geq 3$,

$$|x_n - x_{n-1}| \leq a^{n-2}|x_2 - x_1|.$$

(b) Conclude from (a) that $\{x_n\}$ is a Cauchy sequence. (Hint: It follows from (a) and the triangle inequality that if $3 \leq m < n$, then

$$\begin{aligned} |x_n - x_m| &\leq (a^{n-2} + \cdots + a^{m-1})|x_2 - x_1| \\ &= \frac{a^{m-1} - a^{n-1}}{1 - a}|x_2 - x_1| \\ &\leq a^{m-1}\frac{|x_2 - x_1|}{1 - a} \\ &= a^m \frac{|x_2 - x_1|}{a(1 - a)}. \end{aligned}$$

Since $0 < a < 1$, $a^m$ can be made arbitrarily small if $m$ is large enough.)

**Problem 5.** Let $x_1$ and $x_2$ be distinct real numbers and for $n \geq 3$ define

$$x_n = \frac{1}{2}(x_{n-1} + x_{n-2}).$$

In other words, every member of the sequence beginning with $x_3$ is the average of the two previous members. Show that $\{x_n\}$ is a Cauchy sequence, and conclude that $\lim_{n\to\infty} x_n$ exists. (Hint: To get an idea, draw a picture and use the fact that the average of two points $a, b \in \mathbb{R}$ is the midpoint of the segment $[a, b]$. To prove the result carefully, note that

$$|x_n - x_{n-1}| = \frac{1}{2}|x_{n-1} - x_{n-2}|$$

and apply the result of Problem 4.)

**Problem 6.** Give an example of a sequence $\{x_n\}$ in $\mathbb{R}$ such that $\inf\{x_n\} = 0$ but $\liminf_{n\to\infty} x_n = 1$.

**Problem 7.** True or false: If $\limsup_{n\to\infty} x_n = 2$, then $x_n > 1.99$ for all $n$ large enough.