Math 360 Homework 6

If you make people think they are thinking, they will love you. But if you really make them think, they will hate you.

Problem 1. Which of the following sets is a vector space with the given addition and scalar multiplication?

(a) The set of all non-negative real numbers with usual addition and multiplication.

(b) The set of all functions $f : [-1, 1] \to \mathbb{R}$ which satisfy $f(0) = 0$, with the usual addition of functions and multiplication of functions by real numbers.

(c) The set of all sequences $x = \{x_n\}$ of real numbers for which $\{nx_n\}$ is a bounded sequence. Addition and scalar multiplication are defined by $x + y = \{x_n + y_n\}$ and $\alpha x = \{\alpha x_n\}$.

Problem 2. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$\|x\|_{\max} = \max(|x_1|, \ldots, |x_n|).$$

Carefully prove that $\| \cdot \|_{\max}$ is in fact a norm.

Problem 3. For this problem, you should remember the fact that the absolute value of every continuous function $f : [a, b] \to \mathbb{R}$ reaches its maximum somewhere in the interval $[a, b]$; that is, there is a point $c \in [a, b]$ such that $|f(x)| \leq |f(c)|$ for all $x \in [a, b]$. In this case we write $|f(c)| = \max_{x\in[a,b]} |f(x)|$. (We will discuss this property and its generalizations later in this course.) Now fix some interval $[a, b]$ and let $C[a, b]$ be the vector space of all continuous functions $f : [a, b] \to \mathbb{R}$. Define

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|.$$

(a) Prove that $\| \cdot \|_\infty$ is in fact a norm on $C[a, b]$.

(b) If $[a, b] = [0, 1]$, $f(x) = x^3$, and $g(x) = x$, what is the distance $\|f - g\|_\infty$?

Problem 4. Two vectors x, y in an inner product space $(V, \langle \cdot, \cdot \rangle)$ are called *orthogonal* if $\langle x, y\rangle = 0$. If $V = C[-1, 1]$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$, show that every even function $f \in V$ is orthogonal to every odd function $g \in V$.

Problem 5. Let $a_1, \ldots, a_n$ be n positive real numbers such that $\sum_{i=1}^{n} a_i^2 \leq 1/n$. Show that $\sum_{i=1}^{n} a_i \leq 1$. (Hint: Use Cauchy-Schwarz inequality.)

Problem 6. Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$. Show that the *parallelogram law* holds:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad \text{for all } x, y \in V.$$

How do you interpret this geometrically in the case $V = \mathbb{R}^2$ with the standard inner product and norm?

Problem 7. Again, let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Suppose there is a constant $M > 0$ such that

$$|\langle x, y \rangle| \leq M \qquad \text{whenever } \|x\| = \|y\| = 1. \qquad (*)$$

Show that

$$|\langle x, y \rangle| \leq Mt^2 \qquad \text{whenever } \|x\| \leq t \text{ and } \|y\| \leq t.$$

(Hint: If $\|x\| = 0$ or $\|y\| = 0$, there is nothing to prove. Otherwise, consider the vectors $\frac{1}{\|x\|}x$ and $\frac{1}{\|y\|}y$ and apply $(*)$ to them.)