Math 360 Homework 7

Q: How's a door different from a set?
A: A door is either open or closed, but a set can be neither.

Note: Unless otherwise stated, the metric on $\mathbb{R}^n$ is assumed to be standard.

Problem 1. Let X be the space of all binary words of length 4, i.e., elements of X are of the form $x = x_1x_2x_3x_4$, where the x_i are either 0 or 1; thus X has $2^4 = 16$ elements. Equip X with the word metric

$$d(x,y) = \sum_{i=1}^{4} |x_i - y_i|.$$

Let $x := 0011 \in X$. Describe the balls $B(x,r)$ for $r = 1,2,3,4,5$.

Problem 2. Let (X,d) be a metric space. Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x,y) := \frac{d(x,y)}{1+d(x,y)}.$$

Show that ρ is a metric on X. Note that the ρ-distance between any two points is less than 1.

Problem 3. Describe the ball $B(x_0,1)$ in

(a) $\mathbb{R}^2$ with $x_0 = (0,0)$ and with the metric $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(b) $\mathbb{R}$ with $x_0 = -1$ and with the metric $\rho(x,y) = \dfrac{|x-y|}{1+|x-y|}$.

Problem 4. Let $U \subset \mathbb{R}$ be non-empty and open. Show that

$$U \times \mathbb{R} = \{(x,y) \in \mathbb{R}^2 : x \in U\}$$

is an open subset of $\mathbb{R}^2$. (Hint: To get the idea, first consider the case where U is an open interval and draw a picture of $U \times \mathbb{R}$. The general case should be easy then.)

Problem 5. Let $A \subset \mathbb{R}^n$ be any non-empty set. For any $r > 0$, let $N_r(A)$ denote the *r-neighborhood* of A, which by definition is the set of all points in $\mathbb{R}^n$ whose distance to some point of A is less than r:

$$N_r(A) := \{y \in \mathbb{R}^n : d(x,y) < r \text{ for some } x \in A\}.$$

Evidently, $A \subset N_r(A)$.

(a) What is $N_r(A)$ if A is a single point $\{x\}$?

(b) If A is the closed segment $\{(x, 0) \in \mathbb{R}^2 : -1 \leq x \leq 1\}$, can you draw a picture of $N_r(A)$? What would be the shape of $N_r(A)$ if A were the closed square $[-1, 1] \times [-1, 1]$ in the plane?

(c) Prove that for any non-empty set $A \subset \mathbb{R}^n$, $N_r(A)$ is always open.

Problem 6. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B = \bigcup_{i=1}^n A_i$, show that $\overline{B} = \bigcup_{i=1}^n \overline{A_i}$.

(b) If $B = \bigcup_{i=1}^\infty A_i$, show that $\overline{B} \supset \bigcup_{i=1}^\infty \overline{A_i}$.

Show by an example that in case (b) the inclusion may be proper (i.e., $\overline{B}$ may actually be bigger than $\bigcup_{i=1}^\infty \overline{A_i}$).

Problem 7. Let $A \subset \mathbb{R}$ satisfies $\partial A = \emptyset$. Prove that $A = \emptyset$ or $A = \mathbb{R}$.

Reading assignment: Have a quart of your favorite Haagen Dazs flavor while taking a bubble bath, then read Boas's *A primer of real functions* from page 21 (4. Metric spaces ...) to page 32 (If we look for ...) and watch what happens.