# MATH 360 Homework 8

*Of course there is no reason why the union of open sets should be open; it is just our department's policy.*

**Problem 1.** For a non-empty subset $A$ of a metric space $M$, determine which of the following statements are true (If you believe it is true, prove it; if not, give a counterexample):

(i) $\text{int}(\overline{A}) = \text{int}(A)$
(ii) $\overline{A} \cap A = A$
(iii) $\overline{\text{int}(A)} = A$
(iv) $\partial(\overline{A}) = \partial A$
(v) If $A$ is open, then $\partial A \subset M \setminus A$.

**Problem 2.** Let $M$ be the set consisting of only two letters $a$ and $b$, and let $d$ be the discrete metric on $M$, i.e.,

$$d(a,b) = 1, \ d(a,a) = d(b,b) = 0.$$

Which sequences $\{x_n\}$ in $M$ are convergent?

**Problem 3.** Let $M$ be the real line $\mathbb{R}$ but instead of the standard metric $d(x,y) = |x-y|$ consider the discrete metric

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

on $M$. Does the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converge in this metric space? Can you determine which sequences $\{x_n\}$ converge in this metric space?

**Problem 4.** Let $A$ be a non-empty set in a metric space $(M,d)$. For a point $x \in M$, define the *distance between $x$ and $A$*, denoted by $d(x,A)$, as the infimum of the set of all distances $d(x,a)$, where $a$ runs through the set $A$:

$$d(x,A) = \inf_{a \in A} d(x,a).$$

Prove that

(i) There exists a sequence $a_n \in A$ such that $d(x,a_n) \to d(x,A)$ as $n \to \infty$.
(ii) $d(x,A) = 0$ if and only if $x$ is in the closure of $A$.
(iii) In general, there may not be a point $a \in A$ with $d(x,A) = d(x,a)$. (Hint: Look at the case $A = D(0,1) \subset \mathbb{R}^2$.)

**Problem 5.** Prove directly from the definition that the open unit ball $D(0,1)$ in $\mathbb{R}^3$ is *not* compact. To do this, you must find a cover of $D(0,1)$ by infinitely many open sets which has no finite subcover.

**Problem 6.** Is it true that the union of any number of compact sets is compact? What about the union of a finite number of compact sets? (Hint: To answer the second question, first consider the union of two compact sets.)

**Problem 7.** Let $K$ be a compact set in a metric space $M$ and let $A \subset K$ be closed. Prove that $A$ is compact. (Hint: For any open cover $\{U_i\}$ of $A$, the collection $\{U_i\} \cup (M \setminus A)$ is an open cover of $K$.)