Math 360 Homework 9

An unbounded set goes off to infinity,
or at least the end of the blackboard.

Problem 1. Decide whether or not each of the following sets is compact:

(i) $A = \{x \in \mathbb{R} : x - x^2 > 0\}$

(ii) $A = \{\frac{1}{n^2+3} : n \in \mathbb{N}\} \cup \{0\}$

(iii) $A = \{x \in \mathbb{R}^n : 1 \leq \|x\| \leq 2\}$

(iv) $A = \bigcup_{n=1}^{\infty} R_n$, where R_n is the rectangle $[\frac{1}{n+1}, \frac{1}{n}] \times [-\frac{1}{n}, \frac{1}{n}]$ in the plane $\mathbb{R}^2$.

(v) A is the surface of a sphere in $\mathbb{R}^3$ with the north and south poles removed.

Problem 2. Let K be any finite set of points in a metric space. Using the definition of compactness (in terms of open covers) show that K is compact.

Problem 3. Let $\{x_n\}$ be a sequence of distinct points in a metric space which converges to some x. Assume $x_n \neq x$ for all n. Show that the set $A = \{x_1, x_2, x_3, \ldots\}$ is not compact.

Problem 4. Let $\{x_n\}$ and x be as in the previous problem, but this time define $A = \{x_1, x_2, x_3, \ldots\} \cup \{x\}$. Show that A is compact. (Hint: For any open cover of A, some open set must contain x and hence the x_n for all large n.)

Problem 5. Decide whether or not each of the following sets is path-connected:

(i) $A = [0, 1[\cup]1, 2]$ in $\mathbb{R}$.

(ii) A is the union of all circles centered at $(0, 0)$ in the plane $\mathbb{R}^2$ whose radii are rational numbers.

(iii) $A = \{x \in \mathbb{R}^3 : 1 \leq \|x\| \leq 2\}$

Problem 6. Let $U \subset \mathbb{R}^2$ be open and path-connected. Remove finitely many points from U and call the resulting set V. Show that V is still path-connected.

Problem 7. Suppose A and B are connected sets in a metric space such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is connected.