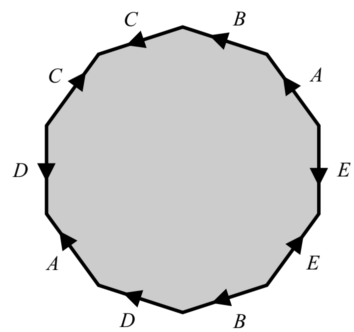
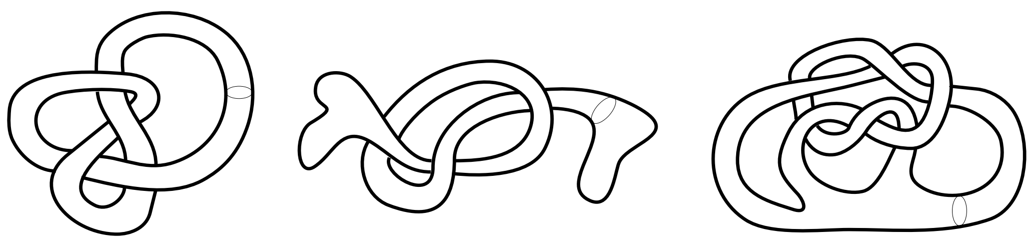
# Math 364 Homework 10 (due 12/6/2002)

**Problem 1.** The edges of a 10-gon in the plane are to be glued as shown to obtain a compact orientable surface $S$. Simplify this polygon as in the proof of the Classification Theorem to reduce it to the standard form. Then determine what kind of surface $S$ is.

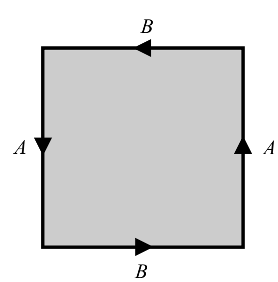



**Problem 2.** Each of the following surfaces is compact and orientable, so it has a well-defined genus $g \geq 0$. In each case, determine this genus.

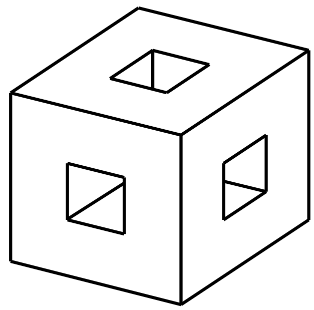


**Problem 3.** Choose a triangulation of the Klein bottle and use it to compute its Euler characteristic.

**Problem 4.** Gluing the edges of a square as shown gives a compact non-orientable surface called the *projective plane*. Choose a triangulation of this surface and use it to compute its Euler characteristic. Compare with the result of problem 3 and conclude that the projective plane is not homeomorphic to the Klein bottle.



**Problem 5.** Drill three holes, each connecting two opposite faces of a cube. What is the genus of the surface of the object obtained this way?



(Hint: Either try to deform this surface into something more familiar, or triangulate it and find the genus by computing its Euler characteristic.)

**Problem 6.** Prove that it is impossible to subdivide the surface of the sphere into a number of regions, each of which is a topological hexagon, such that distinct regions have no more than one edge in common. (Hint: Assuming there is such a subdivision with $v$ vertices, $e$ edges and $f$ faces, check that $e = 3f$ and $v \leq 2f$ and use the Euler characteristic to get a contradiction.)

**Problem 7.** (Optional bonus problem) Suppose $S$ is a compact orientable surface with Euler characteristic $\chi$. Consider any standard triangulation of $S$ (in which two triangles are either disjoint, or meet at one vertex, or meet along one edge). Let $v$ be the number of vertices, $e$ the number of edges and $f$ the number of faces in this triangulation.

(i) Show that
$$3f = 2e.$$

(ii) Define the *degree* of the $i$-th vertex as the number $d_i$ of edges coming together at that vertex. Show that
$$\sum_{i=1}^{v} d_i = 2e.$$

(iii) Using (ii), verify that
$$2e \leq v(v-1).$$

(iv) Using (i), (iii), and the fact that $\chi = v - e + f$, show that
$$v \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi}).$$

(v) The inequality in (iv) says that the number of vertices is at least 4 in the case of the sphere ($\chi = 2$) and 7 in the case of a torus ($\chi = 0$). Are these lower bounds sharp? That is, can you find standard triangulations of the sphere and the torus with 4 and 7 vertices, respectively?