Math 364 Homework 2 (due 9/19/2002)

Problem 1. Show that every finite set of points in $\mathbb{R}^n$ is closed.

Problem 2. Give an example of a closed subset of $\mathbb{R}^3$ which becomes an open set when one of its points is deleted.

Problem 3. Label each of the following sets as open, closed, both, or neither:

- The set of all points x in $\mathbb{R}$ such that $0 < x^2 \leq 1$.
- The parabola $y = x^2$ in $\mathbb{R}^2$
- The set of all points in $\mathbb{R}^2$ whose distance to some point of the parabola $y = x^2$ is less than 0.01.
- The surface of a sphere in $\mathbb{R}^3$ with the north and south poles removed.

Problem 4. Let A and B be subsets of $\mathbb{R}^n$. The *difference set* $A \setminus B$ is the set of all points in A that are not in B (some people use the notation $A - B$ for this set). Thus, for example, A^c (the complement of A) can be described as $\mathbb{R}^n \setminus A$.

(i) Check that $A \setminus B = A \cap B^c$.
(ii) Suppose A is open and B is closed. Show that $A \setminus B$ is open and $B \setminus A$ is closed.

Problem 5. True or false: "If A and $A \cup B$ are open, then so is B" ?

Problem 6. Think of the real line $\mathbb{R}$ as the horizontal axis in the plane $\mathbb{R}^2$; this way every subset of $\mathbb{R}$ can be considered a subset of $\mathbb{R}^2$ as well. Suppose $A \subset \mathbb{R}$ is closed in $\mathbb{R}$. Prove that A is closed in $\mathbb{R}^2$.

Problem 7. Let $A \subset \mathbb{R}^n$. A point p is called an *interior point* of A if there is a ball $B(p, r)$ around it that is contained in A. The set of all the interior points of A (if any) is called *the interior of* A and is denoted by $\text{int}(A)$. Note that $\text{int}(A) \subset A$.

(i) What is $\text{int}(A)$ in each of the following cases?
  - $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$
  - $A = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$
  - $A = \{x \in \mathbb{R}^2 : \|x\| = 1\}$
  - $A = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 1 \text{ or } x_1 \leq -1 \text{ or } x_2 = 0\}$

(ii) Show that for any set A, $\text{int}(A)$ is an open set.
(iii) Show that a set A is open if and only if $A = \text{int}(A)$.
(iv) If U is an open subset of A, show that $U \subset \text{int}(A)$. Roughly speaking, this says that $\text{int}(A)$ is the "largest" open subset of A.