Math 364 Homework 3 (due 9/27/2002)

Problem 1. In each of the following cases, determine whether or not the set A is (relatively) open in the set X:

- X is the interval $(-2, 2)$ in $\mathbb{R}$ and A is the sub-interval $(-2, 1]$.
- X is the closed disk $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ and A is the set of all points in X with positive first coordinate.
- X is the sphere $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ and A is the set of all points in X which are closer to the north pole $(0, 0, 1)$ than to the south pole $(0, 0, -1)$.

Problem 2. Let $X \subset \mathbb{R}^n$, and suppose $U \subset \mathbb{R}^n$ is open. Show that $U \cap X$ is (relatively) open in X.

Problem 3. Recall that for a map $f : X \to Y$, the *image* of a set $A \subset X$ is defined by

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\},$$

while the *preimage* of a set $A \subset Y$ is defined by

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

(i) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = |x_1|$. What is the image of the closed unit disk $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$? What is the preimage of a single point $x \in \mathbb{R}$? (Hint: For the 2nd question, consider the cases $x > 0$, $x = 0$, and $x < 0$ separately.)

(ii) Given a map $f : X \to Y$ and a set $A \subset X$, show that $A \subset f^{-1}(f(A))$. Give an example showing that this is not necessarily an equality; i.e., A may actually be smaller than $f^{-1}(f(A))$.

Problem 4. Suppose $f : X \to Y$ is a map and $A \subset Y$. Show that

$$X \setminus f^{-1}(A) = f^{-1}(Y \setminus A).$$

(Thus, *the complement of the preimage is the preimage of the complement.*) Show by an example that a similar statement is false for images; for instance, find a map $f : \mathbb{R} \to \mathbb{R}$ and a set $A \subset \mathbb{R}$ such that $f(\mathbb{R} \setminus A)$ is different from $f(\mathbb{R}) \setminus f(A)$.

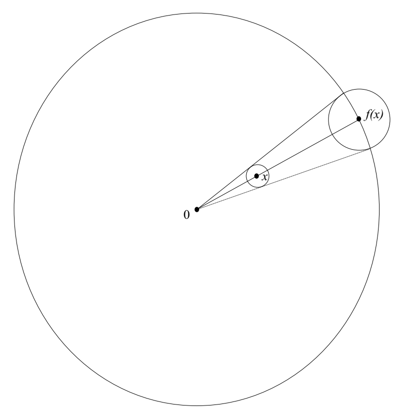
Problem 5. Each of the following maps is discontinuous. In each case, find an open set in the target space whose preimage is not open in the domain:

- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x]$ (the integer part of x).
- $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = \begin{cases} +1 & \text{if } x_2 \geq 0 \\ -1 & \text{if } x_2 < 0 \end{cases}$.
- $f : \mathbb{R} \to \mathbb{R}^3$ defined by $f(x) = \begin{cases} (\cos x, \sin x, +1) & \text{if } x \geq 0 \\ (\cos x, \sin x, -1) & \text{if } x < 0 \end{cases}$.

Problem 6. Consider the "punctured disk" $X = \{x \in \mathbb{R}^2 : 0 < \|x\| < 1\}$ and the unit circle $Y = \{x \in \mathbb{R}^2 : \|x\| = 1\}$. Define $f : X \to Y$ by

$$f(x) = \frac{x}{\|x\|}.$$

Geometrically, $f(x)$ is the "radial projection" of x on the unit circle. Show, using the ε-δ definition, that f is a continuous map. (Hint: In the figure, what is the relation between the radii of the two balls?)



Problem 7. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map which decreases the distances, in the sense that

$$d(f(x), f(y)) \leq d(x, y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

Show that f is necessarily continuous. (Hint: Fix an arbitrary $p \in \mathbb{R}^n$ and check continuity at p using the ε-δ definition.)