Math 364 Homework 5 (due 10/11/2002)

Problem 1. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - 1)^2 = 1\}$ be the unit sphere centered at $(0, 0, 1)$. Recall that a *parallel* in S is a circle obtained by intersecting S with a horizontal plane $z =$ constant, while a *meridian* in S is a circle obtained by intersecting S with a vertical plane containing the z-axis. Consider the stereographic projection $f : S \smallsetminus \{(0, 0, 2)\} \to \mathbb{R}^2$ which has the explicit formula

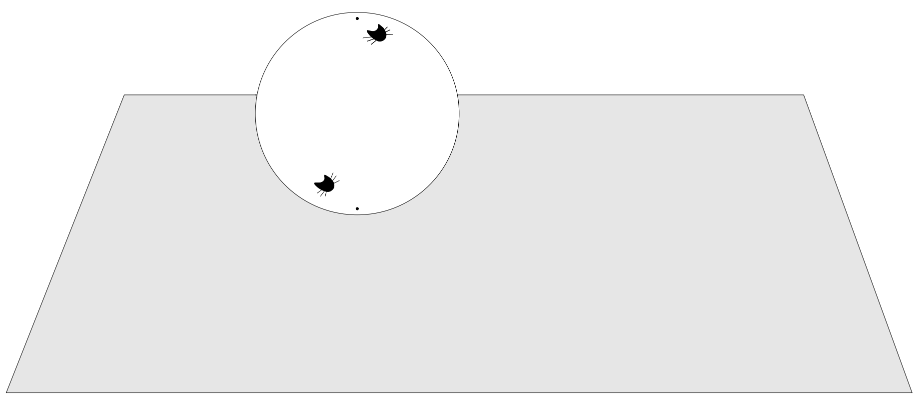
$$f(x, y, z) = \left(\frac{2x}{2 - z}, \frac{2y}{2 - z} \right).$$

Verify that

(i) the image of a parallel in S is a circle in $\mathbb{R}^2$ centered at the origin. What is the relation between the radii of P and its image?

(ii) the image of a meridian in S is a line in $\mathbb{R}^2$ passing through the origin.

Problem 2. Draw the images of the two cats shown on the sphere under the stereographic projection (just try to get the qualitative features right!). What difference between the two images do you notice?



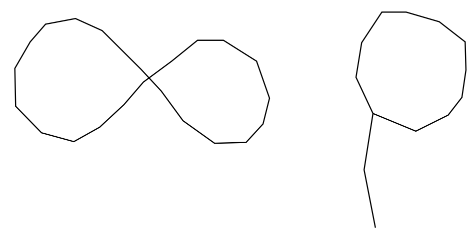
Problem 3. Decide whether or not each of the following sets is path-connected; in each case give reasons for your decision:

- $A = \{x \in \mathbb{R} : 0 < x^2 < 1\}$
- $B = \{(x, y) \in \mathbb{R}^2 : \text{either } x \text{ or } y \text{ is an integer}\}$
- $C = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 2\}$
- $D = \{(x, y, z) \in \mathbb{R}^3 : |z| = x^2 + y^2 + 1\}$.

Problem 4. Let X and Y be subsets of $\mathbb{R}^n$ such that $X \cap Y \neq \emptyset$. Suppose both X and Y are path-connected. Prove that the union $X \cup Y$ must be path-connected. Is the intersection $X \cap Y$ necessarily path-connected?

Problem 5. Show that the union of the two coordinate axes in $\mathbb{R}^2$ is not homeomorphic to the real line $\mathbb{R}$.

Problem 6. Are the following sets homeomorphic? (Assume that the set on the right does not contain the tip of its arm.)



Problem 7. Is there a path-connected set $X \subset \mathbb{R}^2$ such that after removing one of its points the resulting set has infinitely many path-connected components?