Introduction to Holomorphic Dynamics: Very Brief Notes

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Lecture 1.

(1) A *Riemann surface* X is a connected complex manifold of dimension 1. This means that X is a connected Hausdorff space, locally homeomorphic to $\mathbb{R}^2$, which is equipped with a *complex structure* $\{(U_i, z_i)\}$. Here each U_i is an open subset of X, the union $\bigcup U_i$ is X, each *local coordinate* $z_i : U_i \xrightarrow{\cong} \mathbb{D}$ is a homeomorphism, and whenever $U_i \cap U_j \neq \emptyset$ the change of coordinate $z_j z_i^{-1} : z_i(U_i \cap U_j) \to z_j(U_i \cap U_j)$ is holomorphic.

In the above definition we have not assumed that X has a countable basis for its topology. But this is in fact true and follows from the existence of a complex structure (Rado's Theorem).

(2) A map $f : X \to Y$ between Riemann surfaces is holomorphic if $w \circ f \circ z^{-1}$ is a holomorphic map for each pair of local coordinates z on X and w on Y for which this composition makes sense. We often denote this composition by $w = f(z)$. A holomorphic map f is called a *biholomorphism* or *conformal isomorphism* if it is a homeomorphism, in which case f^{-1} is automatically holomorphic.

(3) Examples of Riemann surfaces: The complex plane $\mathbb{C}$, the unit disk $\mathbb{D}$, the Riemann sphere $\widehat{\mathbb{C}}$, complex tori $\mathbb{T}_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ with $\mathrm{Im}(\tau) > 0$, open connected subsets of Riemann surfaces such as the complement of a Cantor set in $\mathbb{C}$.

(4) The Uniformization Theorem: Every simply connected Riemann surface is conformally isomorphic to $\widehat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$.

(5) Classical form of Schwarz Lemma: If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$, then $|f'(0)| \leq 1$. If $|f'(0)| = 1$, then f is a rigid rotation around the origin.

(6) If $f : \mathbb{D}(p, \delta) \to \mathbb{D}(q, \varepsilon)$ is holomorphic, then $|f'(p)| \leq \frac{\varepsilon}{\delta}$.

(7) Corollary (Liouville): Every bounded holomorphic function $\mathbb{C} \to \mathbb{C}$ must be constant.

Lecture 2.

(1) The *automorphism group* of a Riemann surface X is the group of all conformal isomorphisms $X \to X$. It is denoted by $\mathrm{Aut}(X)$.

(2) Theorem:

$$\text{Aut}(\widehat{\mathbb{C}}) = \{z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0\} \cong \text{PSL}_2(\mathbb{C})$$

In particular, $\text{Aut}(\widehat{\mathbb{C}})$ is a connected complex Lie group of dimension 3, homeomorphic to the product $\mathbb{RP}^3 \times \mathbb{R}^3$ (this follows, for example, from the Iwasawa Decomposition).

(3) Theorem:

$$\text{Aut}(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C} \text{ and } a \neq 0\}$$

Thus, $\text{Aut}(\mathbb{C})$ can be identified with the subgroup of $\text{Aut}(\widehat{\mathbb{C}})$ consisting of the maps which fix the point at infinity. It follows that $\text{Aut}(\mathbb{C})$ is a connected complex Lie group of dimension 2, homeomorphic to the product $\mathbb{C}^* \times \mathbb{C}$.

(4) Theorem:

$$\text{Aut}(\mathbb{D}) = \{z \mapsto \lambda \left(\frac{z-a}{1-\overline{a}z}\right) : a \in \mathbb{D} \text{ and } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1\}$$

Thus, $\text{Aut}(\mathbb{D})$ can be identified with the identity component of the subgroup of $\text{Aut}(\widehat{\mathbb{C}})$ consisting of the maps which commute with the reflection $z \mapsto \frac{1}{\overline{z}}$. In particular, $\text{Aut}(\mathbb{D})$ is a connected real Lie group of dimension 3, homeomorphic to the product $\mathbb{D} \times \mathbb{S}^1$.

(5) Theorem:

$$\text{Aut}(\mathbb{H}) = \{z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0\} \cong \text{PSL}_2(\mathbb{R})$$

Thus, $\text{Aut}(\mathbb{H})$ can be identified with the identity component of the subgroup of $\text{Aut}(\widehat{\mathbb{C}})$ consisting of the maps which commute with the reflection $z \mapsto \overline{z}$.

(6) The action of $\text{Aut}(\widehat{\mathbb{C}})$ on $\widehat{\mathbb{C}}$ is simply 3-transitive. Similarly, the action of $\text{Aut}(\mathbb{C})$ on $\mathbb{C}$ is simply 2-transitive. The action of $\text{Aut}(\mathbb{D})$ on $\mathbb{D}$ is transitive but not simply transitive.

(7) Every non-identity $\sigma \in \text{Aut}(\widehat{\mathbb{C}})$ has two fixed points counting multiplicities. If σ has a double fixed point, it can be conjugated to the translation $z \mapsto z + 1$. In this case we call it *parabolic*. If σ has two distinct fixed points, it can be conjugated to the linear map $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. The pair $\{\lambda, \lambda^{-1}\}$ is uniquely determined by σ. We call σ *elliptic* if $|\lambda| = 1$, *hyperbolic* if $\lambda \in \mathbb{R}$ and $|\lambda| \neq 1$, and *loxodromic* otherwise.

(8) An element $\sigma \in \text{Aut}(\widehat{\mathbb{C}})$ can be thought of as a matrix in $\text{PSL}_2(\mathbb{C})$, so $\tau = \text{tr}^2(\sigma)$ is well-defined. Then, σ is parabolic if $\tau = 4$, elliptic if $\tau \in [0, 4[$, hyperbolic if $\tau \in]4, +\infty[$, and loxodromic if $\tau \in \mathbb{C} \setminus [0, +\infty[$.

(9) Let σ, ν be non-identity elements in $\mathrm{Aut}(\widehat{\mathbb{C}})$. If $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\nu)$, then $\sigma\nu = \nu\sigma$. Conversely, if $\sigma\nu = \nu\sigma$, then $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\nu)$ unless σ and ν are involutions, each interchanging the two fixed points of the other (such as the commuting pair $\sigma(z) = -z$ and $\nu(z) = \frac{1}{z}$).

(10) Corollary: Two non-identity elements of $\mathrm{Aut}(\mathbb{C})$ or $\mathrm{Aut}(\mathbb{D})$ commute if and only if they have the same fixed point set.

Lecture 3.

(1) Review of covering space theory:

- Let X be a connected finite dimensional manifold. There exists a covering space $\pi : \widetilde{X} \to X$, with $\widetilde{X}$ simply-connected, called the *universal covering* of X. It is unique up to isomorphism of coverings.
- The *deck group* of $\pi : \widetilde{X} \to X$, denoted by Γ_X, consists of all homeomorphisms $\gamma : \widetilde{X} \to \widetilde{X}$ which satisfy $\pi\gamma = \pi$. Algebraically, Γ_X is isomorphic to the fundamental group $\pi_1(X)$. Once a base point $x \in X$ is chosen, an isomorphism between Γ_X and $\pi_1(X, x)$ can be defined by sending $\gamma \in \Gamma_X$ to the homotopy class of the projection of any path joining some $\tilde{x} \in \pi^{-1}(x)$ to $\gamma(\tilde{x})$.
- Γ_X acts *simply transitively* on the fibers of π: If $\tilde{x}, \tilde{y} \in \widetilde{X}$ with $\pi(\tilde{x}) = \pi(\tilde{y})$, there exists a unique $\gamma \in \Gamma_X$ such that $\gamma(\tilde{x}) = \tilde{y}$. In particular, if $\gamma \in \Gamma_X$ has a fixed point, then $\gamma = \mathrm{id}$.
- Γ_X acts *evenly* on $\widetilde{X}$: Every point in $\widetilde{X}$ has a neighborhood U such that $\gamma(U) \cap U = \emptyset$ for all $\gamma \in \Gamma_X \smallsetminus \{\mathrm{id}\}$. In particular, Γ_X is a discrete subgroup of the homeomorphism group of $\widetilde{X}$. The quotient $\widetilde{X}/\Gamma_X$ is a Hausdorff manifold homeomorphic to X.

There is a one-to-one correspondence between subgroups of Γ_X and coverings of X as follows.

- Given a subgroup $H \subset \Gamma_X$, the quotient $Y = \widetilde{X}/H$ is a covering of X, with the covering map $p : Y \to X$ defined by sending the H-orbit of $\tilde{x} \in \widetilde{X}$ to the Γ_X-orbit of $\tilde{x}$. For this covering, $\pi_1(Y) \cong H$ and the projection $\widetilde{X} \to Y$ is the universal covering. The deck group of $p : Y \to X$ is isomorphic to $N(H)/H$, where

$$N(H) = \{\gamma \in \Gamma_X : \gamma H \gamma^{-1} \subset H\}$$

is the normalizer of H in Γ_X.

- Conversely, given any covering $p : Y \to X$, there exists a covering map $q : \widetilde{X} \to Y$ such that $pq = \pi$. Moreover, there exists a subgroup $H \subset \Gamma_X$ isomorphic to $\pi_1(Y)$ such that $\widetilde{X}/H$ is homeomorphic to Y.

(2) If X is a Riemann surface, the topological universal covering $\pi : \widetilde{X} \to X$ can be equipped with the pull-back complex structure so as to make $\widetilde{X}$ into a Riemann surface, π into a holomorphic map, and Γ_X into a subgroup of $\mathrm{Aut}(\widetilde{X})$.

(3) Corollary: Every Riemann surface X can be represented as $\widetilde{X}/\Gamma$, where $\widetilde{X}$ is conformally isomorphic to $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$, and Γ is a subgroup of $\mathrm{Aut}(\widetilde{X})$ isomorphic to $\pi_1(X)$ which acts evenly on $\widetilde{X}$.

(4) A Riemann surface X is called *spherical*, *Euclidean*, or *hyperbolic* according as its universal covering $\widetilde{X}$ is conformally isomorphic to $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$.

- $\widetilde{X} \cong \widehat{\mathbb{C}}$. Since every automorphism of $\widehat{\mathbb{C}}$ has a fixed point, the only subgroup of $\mathrm{Aut}(\widehat{\mathbb{C}})$ which acts evenly on $\widehat{\mathbb{C}}$ is the trivial group. It follows that $X \cong \widehat{\mathbb{C}}$.
- $\widetilde{X} \cong \mathbb{C}$. The only fixed point free automorphisms of $\mathbb{C}$ are translations. It easily follows that the only subgroups of $\mathrm{Aut}(\mathbb{C})$ which act evenly are the trivial group, or the group generated by a single translation $z \mapsto z + b$, or the group generated by two translations $z \mapsto z + b_1$ and $z \mapsto z + b_2$, with $\frac{b_1}{b_2} \notin \mathbb{R}$. It follows that $X \cong \mathbb{C}$, or $X \cong \mathbb{C}^*$, or $X \cong$ a complex torus.
- $\widetilde{X} \cong \mathbb{D}$. All other Riemann surfaces are therefore in this category. In particular, a Riemann surface with non-abelian fundamental group must be hyperbolic.

(5) Examples: The punctured disk $\mathbb{D}^*$ and the annuli $\mathbb{A}(1, R) = \{z : 1 < |z| < R\}$ are hyperbolic. In fact, these are the only hyperbolic Riemann surfaces with non-trivial abelian fundamental group. The trice punctured sphere $\widehat{\mathbb{C}} \setminus \{a, b, c\}$ is hyperbolic since its fundamental group is non-abelian. If we assume $\{a, b, c\} = \{0, 1, \infty\}$ (a normalization which can always be achieved by applying an automorphism of $\widehat{\mathbb{C}}$), an explicit universal covering map is given by the *elliptic modular function* $\mathbb{H} \to \mathbb{C} \setminus \{0, 1\}$.

(6) Let $f : X \to Y$ be a holomorphic map between Riemann surfaces. If X is non-hyperbolic and Y is hyperbolic, then f is constant.

(7) Corollary (Picard): An entire function which omits two distinct values is constant.

(8) Corollary: A domain $X \subset \widehat{\mathbb{C}}$ is hyperbolic iff $\widehat{\mathbb{C}} \setminus X$ has at least three points.

Lecture 4.

(1) Let $p \in \mathbb{D}$, $\mathbf{v} \in T_p\mathbb{D}$ and choose $\varphi \in \mathrm{Aut}(\mathbb{D})$ so that $\varphi(p) = 0$. Define

$$\|\mathbf{v}\| = 2\,|\varphi_*\mathbf{v}|$$

This is independent of the choice of φ since by Schwarz Lemma every $\psi \in \mathrm{Aut}(\mathbb{D})$ with $\psi(p) = 0$ coincides with φ up to a rotation. Explicitly, take $\varphi(z) = \frac{z-p}{1-\bar{p}z}$ so that

$$\varphi_*\mathbf{v} = \varphi'(p)\,\mathbf{v} = \frac{\mathbf{v}}{1-|p|^2}.$$

Then

$$\|\mathbf{v}\| = \frac{2}{1-|p|^2}\,|\mathbf{v}|.$$

We write this metric as

$$\rho_{\mathbb{D}} = \frac{2}{1-|z|^2}\,|dz|.$$

and call it the *hyperbolic* or *Poincaré metric* of the disk.

Pulling $\rho_{\mathbb{D}}$ back by the conformal isomorphism $f : \mathbb{H} \to \mathbb{D}$ defined by

$$f(z) = \frac{i-z}{i+z},$$

we obtain the following formula for the hyperbolic metric of $\mathbb{H}$:

$$\rho_{\mathbb{H}} = \frac{1}{\mathrm{Im}(z)}\,|dz|.$$

The Gaussian curvature of $\rho_{\mathbb{H}}$ at $z = x + iy \in \mathbb{H}$ can be computed as

$$-\frac{\Delta \log \rho_{\mathbb{H}}(z)}{\rho^2_{\mathbb{H}}(z)} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log y = y^2 \cdot \frac{-1}{y^2} = -1$$

Since the curvature is a conformal invariant, the same holds for $\rho_{\mathbb{D}}$.

(2) Corollary: There exists a smooth Riemannian metric on the unit disk which is invariant under the action of $\mathrm{Aut}(\mathbb{D})$. It is unique up to multiplication by a positive constant, which can be chosen so as to normalize the Gaussian curvature of this metric to -1.

(3) Here are some properties of $\rho_{\mathbb{D}}$. We use the notations $\mathrm{dist}_{\mathbb{D}}(\cdot,\cdot)$ and $B_{\mathbb{D}}(p,r)$ for the hyperbolic distance and the hyperbolic ball centered at p of radius $r > 0$. The same notations without the subscript $\mathbb{D}$ will denote the Euclidean data.

- $\rho_{\mathbb{D}}$ is a *conformal* metric, i.e., at every point it is a positive multiple of the Euclidean metric.
- $\rho_{\mathbb{D}}(z) \to +\infty$ as $|z| \to 1$. In fact, $\rho_{\mathbb{D}}(z)$ is asymptotic to $\frac{1}{\mathrm{dist}(z,\partial\mathbb{D})}$ as $z \to \partial\mathbb{D}$.

- Any two points $p, q \in \mathbb{D}$ can be joined by a unique minimal geodesic. This geodesic is part of the Euclidean circle passing through p, q which is orthogonal to $\partial\mathbb{D}$.
- We have
$$\mathrm{dist}_{\mathbb{D}}(0, z) = \log\left(\frac{1+|z|}{1-|z|}\right) \qquad \text{for all } z \in \mathbb{D}.$$
It follows that
$$B_{\mathbb{D}}(0, r) = B\left(0, \tanh\left(\frac{r}{2}\right)\right)$$
By applying elements of $\mathrm{Aut}(\mathbb{D})$, we conclude that every hyperbolic ball is a Euclidean ball, perhaps with a different center.
- Closed balls in $(\mathbb{D}, \mathrm{dist}_{\mathbb{D}})$ are compact. Hence $(\mathbb{D}, \mathrm{dist}_{\mathbb{D}})$ is a complete metric space.

(4) Let X be a hyperbolic Riemann surface and $\pi : \mathbb{D} \to X$ be its universal covering. The hyperbolic metric $\rho_{\mathbb{D}}$ is invariant under the action of the deck group $\Gamma_X \subset \mathrm{Aut}(\mathbb{D})$, so it descends to a well-defined Riemannian metric ρ_X on X. In local coordinates $w = \pi(z)$, the metric $\rho_X = \rho_X(w)\,|dw|$ satisfies
$$\rho_X(\pi(z)) = \frac{\rho_{\mathbb{D}}(z)}{|\pi'(z)|} = \frac{2}{(1-|z|^2)\,|\pi'(z)|}$$
Clearly, this metric on X makes π into a local isometry.

(5) Some properties of ρ_X:
- ρ_X is a conformal metric of constant curvature -1.
- Closed balls in (X, dist_X) are compact. Hence (X, dist_X) is a complete metric space.
- Geodesics in X are the π-images of geodesics in $\mathbb{D}$.
- Any pair $p, q \in X$ can be joined by at least one minimal geodesic, obtained as follows: Choose $\tilde{p} \in \pi^{-1}(p)$ and $\tilde{q} \in \pi^{-1}(q)$ so that
$$\mathrm{dist}_{\mathbb{D}}(\tilde{p}, \tilde{q}) = \mathrm{dist}_{\mathbb{D}}(\pi^{-1}(p), \pi^{-1}(q)).$$
Then, the π-image of the geodesic joining $\tilde{p}$ to $\tilde{q}$ is a minimal geodesic joining p to q. In particular,
$$\mathrm{dist}_X(p, q) = \mathrm{dist}_{\mathbb{D}}(\pi^{-1}(p), \pi^{-1}(q)).$$

(6) Example: Using the universal covering map $\pi : \mathbb{H} \to \mathbb{D}^*$ given by $\pi(z) = e^{2\pi i z}$, we find that the hyperbolic metric on $\mathbb{D}^*$ has the form
$$\rho_{\mathbb{D}^*} = \frac{-1}{|z|\,\log|z|}\,|dz|.$$

A neighborhood of the cusp in $\mathbb{D}^*$ can be embedded isometrically in $\mathbb{R}^3$ as the *pseudo-sphere*, i.e., the surface obtained by revolving the *tractrix* about its axis.

Lecture 5.

(1) Let $f : X \to Y$ be a holomorphic map between hyperbolic Riemann surfaces. We define

$$\|f'(z)\| = \frac{\rho_Y(f(z))}{\rho_X(z)}\,|f'(z)|.$$

To emphasize the metrics used in the domain and range, we sometimes use the more descriptive notation $\|f'(z)\|_{\rho_X,\rho_Y}$, or $\|f'(z)\|_{\rho_X}$ when $X = Y$. Note that unlike $|f'(z)|$ which depends on the choice of the coordinates, the norm $\|f'(z)\|$ is a well-defined function on X. Clearly, f is a local isometry iff $\|f'(z)\| = 1$ for all $z \in X$. Note also that

$$\|(g \circ f)'(z)\| = \|g'(f(z))\| \cdot \|f'(z)\|$$

(2) Example: Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then $\|\varphi'(z)\| = 1$ for all $z \in \mathbb{D}$ iff $\varphi \in \text{Aut}(\mathbb{D})$.

(3) If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then $\|f'(z)\| \le 1$ for all $z \in \mathbb{D}$. If equality holds at some z, then it holds everywhere and $f \in \text{Aut}(\mathbb{D})$.

(4) Corollary: If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2} \qquad \text{for all } z \in \mathbb{D}$$

(5) Invariant form of Schwarz Lemma: Let $f : X \to Y$ be a holomorphic map between hyperbolic Riemann surfaces. Then $\|f'(z)\| \le 1$ for all $z \in X$. Moreover, exactly one of the following must be the case:

- The equality $\|f'(z)\| = 1$ holds for all z, f is a local isometry and a covering map.
- The strict inequality $\|f'(z)\| < 1$ holds for all z, f is a contraction and not a covering map. In this case, for every compact set $K \subset X$ there exists a constant $0 < c = c(K) < 1$ such that

$$\text{dist}_Y(f(z), f(w)) \le c\ \text{dist}_X(z, w) \qquad \text{for all } z, w \in K$$

(6) Example: $f : \mathbb{D} \to \mathbb{D}$ defined by $f(z) = z^n$ is not a covering map, so it must satisfy $\|f'(z)\| < 1$ for all $z \in \mathbb{D}$. In fact,

$$\|f'(z)\| = \frac{n|z|^{n-1}(1 - |z|^2)}{1 - |z|^{2n}} < 1.$$

Note however that $\|f'(z)\| \to 1$ as $|z| \to 1$. On the other hand, the same f viewed as a map $\mathbb{D}^* \to \mathbb{D}^*$ is a covering, hence a local isometry. This time

$$\|f'(z)\| = \frac{n|z|^{n-1}|z|\log|z|}{|z|^n \log|z|^n} = 1.$$

(7) Corollary: If $X \subsetneq Y$ are hyperbolic Riemann surfaces, the inclusion $\iota : X \hookrightarrow Y$ satisfies $\|\iota'(z)\| = \frac{\rho_Y(z)}{\rho_X(z)} < 1$ for all $z \in X$. In particular,

$$\mathrm{dist}_Y(z,w) < \mathrm{dist}_X(z,w) \qquad \text{for all } z, w \in X.$$

Note that $\|\iota'(z)\| \to 0$ as $z \to \partial X \cap Y$.

Lecture 6.

(1) Let X and Y be Riemann surfaces. Denote by $\mathrm{C}(X,Y)$ the space of all continuous maps from X to Y. This space is endowed with the *compact-open topology* $\mathcal{T}$. A basis for $\mathcal{T}$ is given by the collection

$$U_{K,O} = \{f \in \mathrm{C}(X,Y) : f(K) \subset O\},$$

where K runs through the compact subsets of X and O runs through the open subsets of Y.

(2) The space $\mathrm{C}(X,Y)$ is in fact metrizable. A metric which induces the topology $\mathcal{T}$ can be constructed as follows. Let $\{K_j\}$ be an *exhaustion* of X by a sequence of compact subsets, that is

$$X = \bigcup_{j=1}^{\infty} K_j \qquad \text{and} \quad K_j \subset \mathrm{int}(K_{j+1}) \quad \text{for } j = 1, 2, 3, \ldots$$

Choose any metric d_Y on Y compatible with its topology. For $f, g \in \mathrm{C}(X,Y)$ define

$$\mathbf{d}_j(f,g) = \min\left\{1, \sup_{z \in K_j} d_Y(f(z), g(z))\right\} \qquad j = 1, 2, 3, \ldots$$

and

$$\mathbf{d}(f,g) = \sum_{j=1}^{\infty} 2^{-j}\, \mathbf{d}_j(f,g).$$

It is easy to check that $\mathbf{d}$ is in fact a metric on $\mathrm{C}(X,Y)$, and that it is compatible with $\mathcal{T}$. Furthermore, $\mathbf{d}(f_n, f) \to 0$ iff $f_n \to f$ uniformly on every compact subset of X. For this reason, $\mathcal{T}$ is also called *the topology of local uniform convergence*. In what follows, by the convergence of a sequence we always mean convergence in this topology, unless otherwise stated.

(3) A sequence $f_n \in \mathrm{C}(X,Y)$ *tends to infinity in* Y if for every pair of compact sets $K \subset X$ and $K' \subset Y$ we have $f_n(K) \cap K' = \emptyset$ for all large n.

The definition is interesting only when Y is non-compact. Note also that tending to infinity depends strongly on the target surface Y. For example, $f_n(z) = z + n$ tends to infinity as a sequence of maps $\mathbb{C} \to \mathbb{C}$, but does not tend to infinity as a sequence of maps $\mathbb{C} \to \widehat{\mathbb{C}}$.

(4) A family $\mathcal{F} \subset \mathrm{C}(X,Y)$ is *normal* if every sequence in $\mathcal{F}$ has either a convergent subsequence or a subsequence which tends to infinity in Y.

Clearly, the second possibility never occurs if Y is compact. Also note that normality is a local property, i.e., $\mathcal{F}$ is normal iff every $p \in X$ has a neighborhood U such that $\mathcal{F}|_U$ is normal.

(5) The problem of deciding whether a given family is normal can be quite difficult. Fortunately, for families of holomorphic maps this problem has a surprisingly neat answer, provided by Montel, which is based on the following lemma. Let $\mathrm{Hol}(X,Y)$ denote the closed subspace of $\mathrm{C}(X,Y)$ consisting of all holomorphic maps $X \to Y$.

Lemma: Let X and Y be hyperbolic Riemann surfaces, and $K \subset X$ and $K' \subset Y$ be compact. Then

$$A_{K,K'} = \{f \in \mathrm{Hol}(X,Y) : f(K) \subset K'\}$$

is a compact subset of $\mathrm{Hol}(X,Y)$.

Lecture 7.

(1) Montel's Theorem: If Y is a hyperbolic Riemann surface, then $\mathrm{Hol}(X,Y)$ is a normal family.

(2) Let $Y \subset \widehat{\mathbb{C}}$ be a hyperbolic domain and $f_n \in \mathrm{Hol}(X,Y)$ tend to infinity in Y. Then there is a $w_0 \in \partial Y$ and a subsequence $\{f_{n_k}\}$ which converges to w_0 in $\mathrm{Hol}(X, \widehat{\mathbb{C}})$.

(3) Classical form of Montel's Theorem: Take three distinct points $a, b, c \in \widehat{\mathbb{C}}$ and let $\mathcal{F}_{a,b,c} \subset \mathrm{Hol}(X, \widehat{\mathbb{C}})$ consist of all f which satisfy $f(X) \subset \widehat{\mathbb{C}} \setminus \{a,b,c\}$. Then $\mathcal{F}_{a,b,c}$ is normal.

(4) Let X be a Riemann surface and $f : X \to X$ be a holomorphic map. The *Fatou set* $F(f)$ consists of all points in X with a neighborhood U such that the family $\{f^{\circ n}|_U : U \to X\}_{n \geq 1}$ is normal. The *Julia set* $J(f)$ is the complement $X \setminus F(f)$.

(5) Examples: Consider the case $X = \widehat{\mathbb{C}}$. Then $J(f) = \partial\mathbb{D}$ if $f(z) = z^n$, and $J(f) = \emptyset$ or a point if f is an automorphism.

(6) Some basic properties:

- $J(f)$ is closed and $F(f)$ is open. Either set can be empty. Every connected component of $F(f)$ is called a *Fatou component* of f.
- $J(f)$, hence $F(f)$, is *totally invariant*, that is

$$z \in J(f) \iff f(z) \in J(f).$$

  As a result, $J(f)$ enjoys a great deal of *self-similarity*: If $z \in J(f)$ is not a critical point of f, then there exist neighborhoods U of z and V of $f(z)$ such that $f : U \to V$ is a conformal isomorphism mapping $U \cap J(f)$ homeomorphically to $V \cap J(f)$.
- For any $k \geq 1$, $J(f) = J(f^{\circ k})$, hence $F(f) = F(f^{\circ k})$.

Recall that the *multiplier* of a p-cycle $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_p = z_0$ is the quantity $\lambda = (f^{\circ p})'(z_0) \in \mathbb{C}$ which is well-defined on a Riemann surface. The cycle is *attracting* if $|\lambda| < 1$, *super-attracting* if $\lambda = 0$, *repelling* if $|\lambda| > 1$ and *indifferent* (or *neutral*) if $|\lambda| = 1$. An indifferent cycle is *rationally indifferent* if λ is a root of unity, and *irrationally indifferent* otherwise. A rationally indifferent cycle is called *parabolic* if no iterate of f is the identity map (thus ∞ is a parabolic fixed point for $f(z) = z + 1$ but not for $f(z) = -z$).

- Every repelling cycle is contained in $J(f)$.
- Every attracting cycle is contained in $F(f)$. More precisely, suppose $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_p = z_0$ is attracting and consider its *basin of attraction* U consisting of all $z \in X$ such that $f^{\circ np}(z) \to z_j$ for some j as $n \to \infty$. Then U is open and $U \subset F(f)$.
- Every parabolic cycle is contained in $J(f)$.

Lecture 8.

We now turn to the case of the Riemann sphere. Throughout, $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ will denote a rational map of degree at least 2.

(1) The Julia set $J(f)$ is non-empty.

(2) If $U \subset \widehat{\mathbb{C}}$ is open and $U \cap J(f) \neq \emptyset$, then the union $\bigcup_{n \geq 0} f^{\circ n}(U)$ misses at most two points.

(3) The Julia set $J(f)$ is nowhere dense, or else $J(f) = \widehat{\mathbb{C}}$.

(4) The *grand orbit* of a point $p \in \widehat{\mathbb{C}}$ is the set

$$GO(p) = \{z \in \widehat{\mathbb{C}} : f^{\circ n}(z) = f^{\circ m}(p) \text{ for some } n, m \geq 0\}.$$

The point p is called *exceptional* if $GO(p)$ is a finite set. The set of all exceptional points of f is denoted by $E(f)$.

As an example, $\infty \in E(f)$ whenever f is a polynomial, and $E(f) = \{0, \infty\}$ whenever $f(z) = z^n$, $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$. The next result shows that these examples are quite general.

(5) The exceptional set $E(f)$ has at most two points, which are super-attracting. If $E(f) \neq \emptyset$, then f is conformally conjugate either to a polynomial or to the map $z \mapsto z^n$.

(6) If $z \in J(f)$ and U is a small neighborhood of z, then $\bigcup_{n \geq 0} f^{\circ n}(U) = \widehat{\mathbb{C}} \setminus E(f)$ (in particular, the union does not depend on z or U).

(7) Corollary: If $z \in J(f)$, the set of iterated preimages of z is dense in $J(f)$.

(8) Corollary: The Julia set $J(f)$ is perfect (i.e., it is compact with no isolated point).

(9) For a generic choice of $z \in J(f)$, the forward orbit of z is dense in $J(f)$.

(10) Either $J(f)$ is connected or it has uncountably many connected components.

Lecture 9.

(1) Let X be a hyperbolic Riemann surface and $f : X \to X$ be holomorphic. Then $J(f) = \emptyset$. In particular, f has no repelling or parabolic cycles.

(2) Theorem: Let X be a hyperbolic Riemann surface and $f : X \to X$ be a holomorphic map. Then exactly one of the following must be the case:

(A) *Attracting.* f has a unique fixed point q and the sequence $\{f^{\circ n}\}$ converges locally uniformly to the constant map $X \to \{q\}$.

(E) *Escape.* The sequence $\{f^{\circ n}\}$ tends to infinity in X.

(F) *Finite order.* There exists a $k \geq 1$ such that $f^{\circ k} = \mathrm{id}_X$.

(I) *Irrational rotation.* X is conformally isomorphic to $\mathbb{D}$, $\mathbb{D}^*$ or an annulus $\mathbb{A}(1, R)$, and f acts as an irrational rotation on it.

Here is the structure of the proof:

- Suppose there exists a single orbit of f which tends to infinity in X. Then, using Schwarz Lemma, we show that $\{f^{\circ n}\}$ must tend to infinity in X. This is the case (E).
- Otherwise, all orbits of f are recurrent, that is, for every $p \in X$ there is a compact set $K \subset X$ and an increasing sequence of positive integers n_j such that $f^{\circ n_j}(p) \in K$. We distinguish two cases:
  - •• $\|f'(z)\| < 1$ for all $z \in X$. Then, using Schwarz Lemma, we show that f has an attracting fixed point q, and all orbits converge locally uniformly to q. This is the case (A).

- •• $\|f'(z)\| = 1$ for all $z \in X$ so that f is a covering map. We distinguish two further cases:
  - • • • $\pi_1(X)$ is abelian. Then X is isomorphic to $\mathbb{D}$, $\mathbb{D}^*$ or some $\mathbb{A}(1, R)$ and f is a rotation on X. Depending on whether this rotation is rational or irrational, this is the case (F) or (I).
  - • • • $\pi_1(X)$ is non-abelian. Then by lifting to the universal covering, using Montel, and observing that the normalizer of the deck group of X must be discrete, we show that two distinct iterates of f must coincide. This is the case (F).

Lecture 10.

(1) Corollary: A holomorphic self-map of a hyperbolic Riemann surface with two or more periodic points must be a finite order automorphism.

(2) Fatou-Sullivan's classification of periodic Fatou components: Let $U = f(U)$ be a fixed Fatou component of $f \in \text{Rat}_d$, $d \geq 2$. Then U is

- the "immediate basin" of an attracting fixed point in U, or
- an "attracting petal" for a parabolic fixed point on ∂U with multiplier $\lambda = 1$, or
- a "Siegel disk," or
- a "Herman ring."

The fact that the last two cases can actually occur follows from the work of Siegel, Arnold and Herman. In the case of an attracting petal, the proof of the above theorem relies on the following result:

(3) Snail Lemma: Let f be a holomorphic map defined in a neighborhood V of the fixed point $0 = f(0)$. Let $\gamma : [0, +\infty[\to V \setminus \{0\}$ be a path such that $\lim_{t \to +\infty} \gamma(t) = 0$ and $f(\gamma(t)) = \gamma(t + 1)$ for all $t \geq 0$. Then $|f'(0)| < 1$ or else $f'(0) = 1$.

(4) The immediate basin of an attracting cycle of a rational map of degree ≥ 2 contains a critical point.

(5) Corollary: Let $f \in \text{Rat}_d$, $d \geq 2$. Then f has at most $2d - 2$ attracting cycles.

In fact, such f has at most $2d - 2$ non-repelling cycles. This was conjectured by Fatou and proved by Shishikura.

Lecture 11.

(1) A map $f : X \to Y$ between topological surfaces is *proper* if the preimage of every compact set is compact. Equivalently, if $\{f(x_n)\}$ tends to infinity in Y whenever $\{x_n\}$ tends to infinity in X.

A proper holomorphic map $f : X \to Y$ between Riemann surfaces has a well-defined finite mapping degree, that is, there exists an integer $d \geq 1$ such that

$$\sum_{x \in f^{-1}(y)} \deg(f, x) = d \quad \text{for all } y \in Y.$$

The integer d is often denoted by $\deg(f)$.

(2) A map $f : X \to Y$ between topological surfaces is a *branched covering* if every $y \in Y$ has a small disk neighborhood V such that the preimage of (V, y) is the disjoint union of pointed disks (U_i, x_i), with $f : (U_i, x_i) \to (V, y)$ acting as a power. This means there are homeomorphisms $\phi : (U_i, x_i) \to (\mathbb{D}, 0)$ and $\psi : (V, y) \to (\mathbb{D}, 0)$ such that $\psi \circ f \circ \phi^{-1}(z) = z^k$ for some integer $k \geq 1$. The integer k is called the *local degree* of f at x_i and is denoted by $\deg(f, x_i)$. It is easy to check that a branched covering has a well-defined mapping degree which may be finite or infinite.

A non-constant holomorphic map between Riemann surfaces is proper if and only if it is a finite degree branched covering.

(3) Corollary: A proper holomorphic map between Riemann surfaces with no critical points is a covering map.

(4) Riemann-Hurwitz Formula: Let $f : X \to Y$ be a non-constant proper holomorphic map between Riemann surfaces. Then

$$\deg(f) \cdot \chi(Y) - \chi(X) = \sum_{x \in X} [\, \deg(f, x) - 1 \,].$$

Note that the right term is a finite sum. It is the number of critical points of f counting multiplicities.

In what follows we consider examples of smooth Julia sets:

(5) A circle: If $f(z) = z^{\pm d}$ where $d \geq 2$, then $J(f) = \mathbb{S}^1 = \partial\mathbb{D}$.

(6) An interval: If f is the quadratic Chebyshev polynomial $z \mapsto z^2 - 2$, then $J(f) = [-2, 2]$. This can be verified directly using the fact that $f^{-1}([-2, 2]) = [-2, 2]$. Alternatively, observe that the degree 2 rational function $j(w) = w + w^{-1}$ (known as the *Joukowski map*) semi-conjugates f to the squaring map $s(w) = w^2$:

$$j \circ s = f \circ j.$$

In other words, f is a *quotient* of s. Since j maps both $\mathbb{D}$ and $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ conformally onto $\widehat{\mathbb{C}} \smallsetminus [-2, 2]$, it follows that $\widehat{\mathbb{C}} \smallsetminus [-2, 2]$ is the basin of infinity of f and $J(f) = [-2, 2]$.

The existence of the semi-conjugacy j has interesting consequences. As an example, the normalized Lebesgue measure $\frac{1}{2\pi} d\theta$ on $\mathbb{S}^1$ is an invariant ergodic measure for the action of s, so the push-forward $\mu = j_*(\frac{1}{2\pi} d\theta)$ is an invariant ergodic

measure for the action of f on its Julia set $[-2,2]$. Since $x = j(e^{i\theta}) = 2\cos(\theta)$, if $[a,b] \subset [-2,2]$, we have

$$j^{-1}[a,b] = [\arccos(\frac{b}{2}), \arccos(\frac{a}{2})] \cup [-\arccos(\frac{a}{2}), -\arccos(\frac{b}{2})].$$

It follows that

$$\mu[a,b] = 2 \cdot \frac{1}{2\pi} \cdot (\arccos(\frac{a}{2}) - \arccos(\frac{b}{2})) = \frac{1}{\pi} \int_a^b \frac{dx}{\sqrt{4-x^2}},$$

which gives

$$\mu = \frac{1}{\pi} \frac{dx}{\sqrt{4-x^2}}.$$

Now, ergodicity of μ implies that for any measurable set $E \subset [-2,2]$, the relation

$$\lim_{n \to +\infty} \frac{1}{n} \#\{0 \le k \le n-1 : f^{\circ k}(x) \in E\} = \mu(E)$$

holds for μ-a.e. (hence Lebesgue a.e.) $x \in [-2,2]$. Since μ is more concentrated near the end points ± 2 than the middle of the interval $[-2,2]$, it follows that a typical orbit in the Julia set of f visits the ends of $[-2,2]$ much more frequent than its middle.

(7) Euclidean circles and intervals are the only examples of 1-dimensional smooth Julia sets. According to D. H. Hamilton, if the Julia set of a rational map is a Jordan curve (resp. arc), then either it is a Euclidean circle (resp. circular arc) or it has Hausdorff dimension > 1.

Lecture 12.

Following Lattès, we now construct rational maps f of degree ≥ 2 with $J(f) = \widehat{\mathbb{C}}$.

(1) Take the lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z} \subset \mathbb{C}$ generated by 1 and some $\tau \in \mathbb{H}$, and consider the torus $\mathbb{T}_\tau = \mathbb{C}/\Lambda$. Let σ be the involution $z \mapsto -z$ on $\mathbb{T}_\tau$. Then the quotient $X = \mathbb{T}_\tau/\sigma$ is a compact orientable topological surface and the canonical projection $\pi : \mathbb{T}_\tau \to X$ is a degree 2 branched covering with branch points at $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ corresponding to the fixed points of σ. It is easy to see that X inherits a Riemann surface structure with respect to which π is holomorphic. By Riemann-Hurwitz Formula,

$$2\,\chi(X) - \chi(\mathbb{T}_\tau) = 4 \Longrightarrow \chi(X) = 2,$$

so X is biholomorphic to the Riemann sphere $\widehat{\mathbb{C}}$. An example of such a projection $\pi : \mathbb{T}_\tau \to \widehat{\mathbb{C}}$ is provided by the classical *Weierstrass $\wp$-function*. This is the unique

meromorphic function on $\mathbb{C}$ with poles on Λ which satisfies

$$\begin{aligned}\wp(z+1) = \wp(z+\tau) &= \wp(z)\\ \wp(-z) &= \wp(z)\\ \wp(z) &= z^{-2} + O(1) \quad \text{near } z = 0.\end{aligned}$$

Explicitly,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \smallsetminus \{0\}} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right].$$

Now consider the endomorphism $g : \mathbb{T}_\tau \to \mathbb{T}_\tau$ defined by $g(z) = nz$ for an integer $n \geq 2$. Then g is a holomorphic covering map of degree n^2 which is uniformly expands the Euclidean metric on $\mathbb{T}_\tau$. Since all points of the form

$$\frac{r}{n^p - 1} + \frac{s}{n^p - 1}\,\tau \qquad (r, s \in \mathbb{Z})$$

are fixed under $g^{\circ p}$, it follows that the periodic orbits of g are dense in $\mathbb{T}_\tau$. Since all these orbits are repelling, we have $J(g) = \mathbb{T}_\tau$. As g commutes with the involution σ, it descends to a well-defined holomorphic map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the same degree n^2. Moreover, $J(f) = \widehat{\mathbb{C}}$ since π maps the repelling cycles of g to repelling cycles of f. We call f constructed this way a *Lattès map*.

(2) Thus, for an arbitrary lattice we can construct Lattès maps of degree ≥ 4. For special lattices, we can obtain Lattès maps of lower degrees. For example, take Λ to be the lattice of Gaussian integers $\mathbb{Z} \oplus i\mathbb{Z}$, and define $g : \mathbb{T}_i \to \mathbb{T}_i$ by $g(z) = (1+i)z$. Then g is a holomorphic expanding map of degree $|1+i|^2 = 2$, and a brief inspection shows that $J(g) = \mathbb{T}_i$. It follows that the induced map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is rational of degree 2 with $J(f) = \widehat{\mathbb{C}}$.

To find an explicit formula for f, normalize the projection π so that $\pi(0) = \infty$, $\pi(\frac{1}{2}) = 1$ and $\pi(\frac{1+i}{2}) = 0$. To figure out what $\pi(\frac{i}{2})$ is, note that $z \mapsto iz$ is an automorphism of $\mathbb{T}_i$ which fixes 0 and $\frac{1+i}{2}$ and interchanges $\frac{1}{2}$ and $\frac{i}{2}$. It descends to an involution of $\widehat{\mathbb{C}}$ which fixes 0 and ∞ and interchanges 1 and $\pi(\frac{i}{2})$. This involution is necessarily $w \mapsto -w$, so $\pi(\frac{i}{2}) = -1$. Thus

$$\begin{array}{rcccccc} \text{critical points of } \pi: & 0 & & \dfrac{1}{2} & \dfrac{i}{2} & & \dfrac{1+i}{2} \\ & \downarrow & & \downarrow & \downarrow & & \downarrow \\ \text{critical value of } \pi: & \infty & & 1 & -1 & & 0 \end{array}$$

Under g, the critical points of π map as

$$\frac{1}{2}, \frac{i}{2} \mapsto \frac{1+i}{2} \mapsto 0 \mapsto 0,$$

so under f, the critical values of π map as

$$\pm 1 \mapsto 0 \mapsto \infty \mapsto \infty.$$

It easily follows that the quadratic rational map f must have the form $f(w) = \lambda(w - w^{-1})$. To determine λ, use $\pi \circ g = f \circ \pi$ and the fact that $\pi(z) \sim z^{-2}$ near $z = 0$ to obtain

$$\frac{1}{(1+i)^2 z^2} \sim \frac{\lambda}{z^2} \quad \text{near } z = 0.$$

Thus $\lambda = \frac{1}{2i}$ and $f(w) = \frac{1}{2i}(w - w^{-1})$. Note in particular that the critical values of π are precisely the *postcritical* points of f (a fact that could have also been checked using $\pi \circ g = f \circ \pi$).

(3) The endomorphism $g(z) = (1+i)z$ of the torus $\mathbb{T}_i$ has an obvious invariant ergodic measure, namely the Lebesgue area form $|dz|^2$. It follows that the push-forward $\mu = \pi_*(|dz|^2)$ is an invariant ergodic measure for the Lattès map $f(w) = \frac{1}{2i}(w - w^{-1})$. To find the explicit formula for this measure, note that the elliptic function $w = \pi(z)$ satisfies the differential equation

$$\left(\frac{dw}{dz}\right)^2 = 4w^3 - g_2 w - g_3,$$

where the constants g_2, g_3 are determined by the fact that $0, \pm 1$ are the finite critical values of w. Thus

$$\left(\frac{dw}{dz}\right)^2 = 4w(w-1)(w+1),$$

so

$$|dz|^2 = \frac{|dw|^2}{4|w|\,|w-1|\,|w+1|}.$$

Since the map π has degree 2, it follows that

$$\mu = \frac{|dw|^2}{2\,|w|\,|w-1|\,|w+1|}.$$

Note that μ is a smooth measure except at the four critical values of π which are the postcritical points of f. Ergodicity of μ shows that a typical orbit of f visits neighborhoods of these four points much more frequently than neighborhoods of the same spherical size of other points on the sphere. Thus, such a typical orbit appears to be highly concentrated near the four postcritical points.

Lecture 13.

(1) Rational maps of degree two or more exhibit both expanding and contracting behaviors. Let $f \in \mathrm{Rat}_d$ with $d \geq 2$. Then $f^{\circ n}$ has $2d^n - 2$ critical points near which

the spherical metric $\sigma = 2|dz|/(1+|z|^2)$ is highly contracted. On the other hand, if $\|f'(z)\|_\sigma = \sigma(f(z))|f'(z)|/\sigma(z)$ is the spherical norm of the derivative, we have

$$\int_{\widehat{\mathbb{C}}} (f^{\circ n})^* \sigma^2 = \int_{\widehat{\mathbb{C}}} \|(f^{\circ n})'\|_\sigma^2 \, \sigma^2 = d^n \int_{\widehat{\mathbb{C}}} \sigma^2 = 4\pi d^n,$$

which shows on average $\|(f^{\circ n})'\|_\sigma$ grows exponentially fast.

(2) Given $f \in \text{Rat}_d$ with $d \geq 2$, let $C(f)$ denote the set of critical points of f. Define the *postcritical set* $P(f)$ as

$$P(f) = \overline{\bigcup_{c \in C(f)} \{f(c), f^{\circ 2}(c), f^{\circ 3}(c), \ldots\}}.$$

Here are some basic properties:

- $P(f)$ is non-empty and compact.
- $f(P(f)) \subset P(f)$.
- $P(f) = P(f^{\circ k})$ for every $k \geq 1$.
- $P(f)$ is the smallest compact subset of $\widehat{\mathbb{C}}$ which contains all the critical values of $f^{\circ n}$ for all $n \geq 1$.

The last property shows that on every topological disk disjoint from $P(f)$, all the d^n inverse branches of $f^{\circ n}$ are single-valued holomorphic functions.

(3) Suppose $f \in \text{Rat}_d$ with $d \geq 2$, and $\#P(f) \leq 2$. Then f is conjugate to the map $z \mapsto z^d$ or $z \mapsto z^{-d}$ (in particular $\#P(f) = 2$).

(4) Let $f \in \text{Rat}_d$ with $d \geq 2$, and $\#P(f) \geq 3$. Let ρ denote the hyperbolic metric on (each component of) the complement $\widehat{\mathbb{C}} \smallsetminus P(f)$. If both z and $f(z)$ are in $\widehat{\mathbb{C}} \smallsetminus P(f)$, then $\|f'(z)\|_\rho \geq 1$.

(5) Let $f \in \text{Rat}_d$ with $d \geq 2$, and $\#P(f) \geq 3$. Define ρ as above, and assume that the forward orbit of $z \in J(f)$ never hits $P(f)$. Then $\|(f^{\circ n})'(z)\|_\rho \to \infty$ as $n \to \infty$.

(6) Corollary: Let $f \in \text{Rat}_d$ with $d \geq 2$. Then $P(f)$ contains the attracting, parabolic, and Cremer cycles, as well as the boundaries of Siegel disks and Herman rings of f.

Lecture 14.

Throughout we assume $f \in \text{Rat}_d$ with $d \geq 2$.

(1) f is *expanding* (on its Julia set) if there exists a conformal metric $\rho = \rho(z)|dz|$ defined in a neighborhood of $J(f)$ and some $\lambda > 1$ such that

$$\|f'(z)\|_\rho > \lambda \qquad \text{for all } z \in J(f).$$

(2) Theorem: The following conditions are equivalent:

(i) f is expanding.

(ii) For *any* conformal metric ρ defined near the Julia set, there exist $C > 0$ and $\lambda > 1$ such that $\|(f^{\circ n})'(z)\|_\rho > C\lambda^n$ for all $n \geq 1$ and all $z \in J(f)$.
(iii) Some finite iterate $f^{\circ k}$ expands the spherical metric σ on the Julia set, i.e., there exists $\lambda > 1$ such that $\|(f^{\circ k})'(z)\|_\sigma > \lambda$ for all $z \in J(f)$.
(iv) There are no critical points of f in $J(f)$ and all orbits in $F(f)$ tend to attracting cycles.
(v) The orbit of every critical point of f tends to an attracting cycle.
(vi) $P(f) \cap J(f) = \emptyset$.

(3) Corollary: An expanding map has no indifferent cycles or Herman rings.

(4) Corollary: A quadratic polynomial with an attracting cycle in the plane is expanding.

(5) A central conjecture in holomorphic dynamics is that expanding rational (resp. polynomial) maps of degree $d \geq 2$ are dense in Rat_d (resp. Pol_d). This would follow from the conjecture that structurally stable rational maps are expanding, since the former class is known to be dense by the work of Mañe-Sad-Sullivan.

Lecture 15.

(1) Classical form of Köebe Distortion Theorem: Let $f : \mathbb{D} \to \mathbb{C}$ be univalent, with $f'(0) = 1$. Then

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} \qquad \text{if } |z| \leq r < 1.$$

It follows that if $f : \mathbb{D}(p, \varepsilon) \to \mathbb{C}$ is univalent and $r < 1$, then the *distortion* of f on the smaller disk $\mathbb{D}(p, r\varepsilon)$ defined by

$$\sup \left\{ \frac{|f'(z)|}{|f'(w)|} : z, w \in \mathbb{D}(p, r\varepsilon) \right\}$$

is bounded by a constant $C(r) > 0$ independent of f. Thus, f distorts arc-lengths and areas in $\mathbb{D}(p, r\varepsilon)$ by a factor which only depends on r.

(2) Invariant form of Köebe Distortion Theorem: Let $U \subsetneq \mathbb{C}$ be a simply-connected domain, $K \subset U$ be compact, and d be the hyperbolic diameter of K in U. Then, for every univalent function $f : U \to \mathbb{C}$,

$$\sup \left\{ \frac{|f'(z)|}{|f'(w)|} : z, w \in K \right\} \leq e^{4d}.$$

(3) Lebesgue Density Theorem: Let $E \subset \mathbb{C}$ be measurable. Then

$$\lim_{r \to 0} \frac{\text{area}(\mathbb{D}(p, r) \cap E)}{\text{area}(\mathbb{D}(p, r))} = \chi_E(p) \quad \text{for almost every } p \in \widehat{\mathbb{C}}.$$

In particular, if $\text{area}(E) > 0$, then almost every $p \in E$ is a *density point*, i.e., it satisfies $\lim_{r \to 0} \text{area}(\mathbb{D}(p, r) \cap E) / \text{area}(\mathbb{D}(p, r)) = 1$.

(4) In the definition of density points, we can replace round disks by other kinds of neighborhoods. For example, a topological disk D is called α-*round* if

$$\operatorname{area}(D) \geq \alpha \operatorname{diam}(D)^2.$$

If p is a density point of E and $\{D_k\}$ is a sequence of α-round neighborhoods of p with $r_k = \operatorname{diam}(D_k) \to 0$, then

$$\begin{aligned}
\frac{\operatorname{area}(D_k \cap E)}{\operatorname{area}(D_k)} &= 1 - \frac{\operatorname{area}(D_k \smallsetminus E)}{\operatorname{area}(D_k)} \\
&\geq 1 - \frac{\operatorname{area}(\mathbb{D}(p, r_k))}{\operatorname{area}(D_k)} \cdot \frac{\operatorname{area}(\mathbb{D}(p, r_k) \smallsetminus E)}{\operatorname{area}(\mathbb{D}(p, r_k))} \\
&\geq 1 - \frac{\pi r_k^2}{\alpha r_k^2} \cdot \frac{\operatorname{area}(\mathbb{D}(p, r_k) \smallsetminus E)}{\operatorname{area}(\mathbb{D}(p, r_k))} \\
&= 1 - \frac{\pi}{\alpha} \cdot \frac{\operatorname{area}(\mathbb{D}(p, r_k) \smallsetminus E)}{\operatorname{area}(\mathbb{D}(p, r_k))}.
\end{aligned}$$

Since $\operatorname{area}(\mathbb{D}(p, r_k) \smallsetminus E) / \operatorname{area}(\mathbb{D}(p, r_k)) \to 0$ as $k \to \infty$, it follows that

$$\lim_{k \to \infty} \frac{\operatorname{area}(D_k \cap E)}{\operatorname{area}(D_k)} = 1.$$

(5) A rational map f is said to be *ergodic* (with respect to the Lebesgue measure class on the sphere) if for every measurable set $E \subset \widehat{\mathbb{C}}$ which satisfies $E = f^{-1}(E)$ it is true that $\operatorname{area}(E) = 0$ or $\operatorname{area}(\widehat{\mathbb{C}} \smallsetminus E) = 0$.

It is not hard to show that $J(f) = \widehat{\mathbb{C}}$ whenever f is ergodic.

(6) "Ergodic or Attracting" Theorem: Suppose $f \in \operatorname{Rat}_d$ with $d \geq 2$. Then

- f is ergodic and hence $J(f) = \widehat{\mathbb{C}}$, or
- $\lim_{n \to \infty} \operatorname{dist}_\sigma(f^{\circ n}(z), P(f)) = 0$ for almost every $z \in J(f)$.

The two possibilities are *not* mutually exclusive: According to M. Rees, there are ergodic rational maps with $P(f) = J(f) = \widehat{\mathbb{C}}$.

(7) Corollary: If f is expanding, then $\operatorname{area}(J(f)) = 0$.

A more careful analysis of the proof shows that in fact $\dim_{\mathrm{H}}(J(f)) < 2$.

Lecture 16.

(1) Let X be a smooth orientable surface. A *conformal structure* on X is an equivalence class of smooth Riemannian metrics, where the metrics g, g' are considered equivalent if $g' = \eta\, g$ for some positive function $\eta : X \to \mathbb{R}$. The equivalence class (also called the *conformal class*) of g is denoted by $[g]$.

It follows from the definition that each conformal structure gives rise to a well-defined notion of "angle" between tangent vectors.

(2) Now assume in addition that X has a complex structure. Then X carries a canonical conformal structure whose representative metrics have the local form $g(z) = \gamma(z)\,|dz|$ in each holomorphic local coordinate z on X. This is well-defined since if ζ is another local coordinate near the same point and $\zeta \mapsto z(\zeta)$ is the change of coordinates, then

$$g(\zeta) = \gamma(z(\zeta))\,|z'(\zeta)|\,|d\zeta|$$

which is a multiple of $|d\zeta|$. We call this conformal structure the *standard conformal structure of* X (with respect to the given complex structure) and denote it by σ_X.

(3) On a Riemann surface, it is often easier to do local computations in complex-variable notations. Let X be a Riemann surface and $z = x + iy$ be a holomorphic local coordinate on X. Then (x, y) can be thought of as coordinates for the underlying smooth surface. In these coordinates, a Riemannian metric g has the local form

$$E\,dx^2 + 2F\,dx\,dy + G\,dy^2,$$

where E, F, G are smooth functions of (x, y) satisfying $E > 0$, $G > 0$ and $EG - F^2 > 0$. The associated inner product on each tangent space is given by

$$\begin{aligned}\left\langle a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\ ,\ c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}\right\rangle &= Eac + F(ad + bc) + Gbd \\ &= [a \quad b]\ L\ \begin{bmatrix} c \\ d \end{bmatrix}\end{aligned}$$

where

$$L = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

is the matrix of g in the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. In particular,

$$\left\| a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right\|^2 = Ea^2 + 2Fab + Gb^2.$$

Define two local sections of the complexified cotangent bundle $T^*X \otimes \mathbb{C}$ by

$$\begin{aligned} dz &= dx + i\,dy \\ d\overline{z} &= dx - i\,dy \end{aligned}$$

which form a basis for each complexified cotangent space. The local sections

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) \end{aligned}$$

of the complexified tangent bundle $TX \otimes \mathbb{C}$ will form the dual basis at each point. The inner product g extends uniquely to a Hermitian product on $TX \otimes \mathbb{C}$. The matrix of this Hermitian product in the basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}\}$ is given by $L' = D^* \, L \, D$, where

$$D = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$

It follows that

$$L' = \frac{1}{4} \begin{bmatrix} E+G & E-G-2iF \\ E-G+2iF & E+G \end{bmatrix}.$$

Let us introduce the quantities

$$\gamma^2 = \frac{1}{4}(E+G+\sqrt{EG-F^2})$$
$$\mu = \frac{1}{4\gamma^2}(E-G+2iF) = \frac{E-G+2iF}{E+G+\sqrt{EG-F^2}}\,.$$

Note that

$$\gamma^2 > 0 \quad \text{and} \quad |\mu|^2 = \frac{E+G-2\sqrt{EG-F^2}}{E+G+2\sqrt{EG-F^2}} < 1.$$

Substituting these into L' gives

$$L' = \gamma^2 \begin{bmatrix} \dfrac{1+|\mu|^2}{2} & \overline{\mu} \\ \mu & \dfrac{1+|\mu|^2}{2} \end{bmatrix}.$$

Since the Hermitian product on $T^*X \otimes \mathbb{C}$ is given by

$$\left\langle \alpha\frac{\partial}{\partial z} + \beta\frac{\partial}{\partial \overline{z}} \,,\, \omega\frac{\partial}{\partial z} + \nu\frac{\partial}{\partial \overline{z}} \right\rangle = [\alpha \quad \beta]\, L' \begin{bmatrix} \overline{\omega} \\ \overline{\nu} \end{bmatrix},$$

it follows that

$$\left\| \alpha\frac{\partial}{\partial z} + \beta\frac{\partial}{\partial \overline{z}} \right\|^2 = \gamma^2 \left(\frac{1+|\mu|^2}{2}\,(|\alpha|^2+|\beta|^2) + \mu\,\overline{\alpha}\,\beta + \overline{\mu}\,\alpha\,\overline{\beta} \right).$$

But *real* tangent vectors have the special form $\alpha\frac{\partial}{\partial z} + \overline{\alpha}\frac{\partial}{\partial \overline{z}}$, since $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} = (a+ib)\frac{\partial}{\partial z} + (a-ib)\frac{\partial}{\partial \overline{z}}$. For such vectors, the above formula reduces to

$$\left\| \alpha\,\frac{\partial}{\partial z} + \overline{\alpha}\,\frac{\partial}{\partial \overline{z}} \right\|^2 = \gamma^2 |\alpha + \mu\,\overline{\alpha}|^2.$$

The last expression suggests that as long as we care about lengths of real tangent vectors, the Hermitian metric g in the complex basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}\}$ can be represented as

$$g = \gamma(z)\,|dz + \mu(z)\,d\overline{z}|,$$

with γ and μ defined as above.

(4) Let us see how the quantities γ and μ associated with a metric g transform under a holomorphic change of coordinates $z \mapsto w$ on X:

$$\begin{aligned}\gamma(z)\,|dz+\mu(z)\,d\overline{z}| &= w^*(\gamma(w)\,|dw+\mu(w)\,d\overline{w}|)\\ &= \gamma(w(z))\,|w'(z)\,dz+\mu(w(z))\overline{w'(z)}\,d\overline{z}|\\ &= \gamma(w(z))\,|w'(z)|\,\left|dz+\mu(w(z))\frac{\overline{w'(z)}}{w'(z)}\,d\overline{z}\right|,\end{aligned}$$

from which we obtain

$$\begin{aligned}\gamma(z) &= \gamma(w(z))\,|w'(z)|\\ \mu(z) &= \mu(w(z))\,\frac{\overline{w'(z)}}{w'(z)}\end{aligned}$$

or simply

$$\begin{aligned}\gamma(z)\,|dz| &= \gamma(w)\,|dw|\\ \mu(z)\,\frac{d\overline{z}}{dz} &= \mu(w)\,\frac{d\overline{w}}{dw}.\end{aligned}$$

It follows that $\gamma(z)\,|dz|$ is a well-defined $(1,1)$-differential, namely a conformal metric, on X. Similarly, $\mu(z)\,\frac{d\overline{z}}{dz}$ is a well-defined $(-1,1)$-differential on X. We call $\mu = \mu(z)\,\frac{d\overline{z}}{dz}$ the *Beltrami differential* associated with g. Note that μ depends only on the conformal class $[g]$. Note also that $z \mapsto |\mu(z)|$ is a well-defined function on X.

(5) Corollary: There is a one-to-one correspondence between conformal structures on a Riemann surface X and Beltrami differentials $\mu = \mu(z)\,\frac{d\overline{z}}{dz}$ which satisfy $|\mu(z)| < 1$ in every local coordinate z on X. The standard conformal structure σ_X corresponds to the zero Beltrami differential.

(6) Here is the geometric interpretation of a Beltrami differential associated with a conformal structure $[g]$: Fix a local coordinate $z = x + iy \cong (x,y)$ near a point $p \in X$. Consider the family of "circles" $E(p) = \{\mathbf{v} \in T_pX : \|\mathbf{v}\| = \text{const.}\}$ which depends only on $[g]$. If $\mathbf{v} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} = (a+ib)\frac{\partial}{\partial z} + (a-ib)\frac{\partial}{\partial \overline{z}}$, then the "circles" $\|\mathbf{v}\| = \text{const.}$ correspond to the loci $|(a+ib)+\mu(a-ib)| = \text{const.}$ in the real (a,b)-plane. Setting $\mu = re^{i\theta}$ and $\zeta = (a+ib)e^{-i\frac{\theta}{2}}$, we obtain the loci $|\zeta + r\overline{\zeta}| = \text{const.}$ in the ζ-plane, which is the family of concentric ellipses with the minor axis along the real direction and the major axis along the imaginary direction, and with the ratio of the major to minor axis equal to $\frac{1+r}{1-r}$. Transferring this family back to the

(a,b)-plane, it follows that $E(p)$ is a family of concentric ellipses in T_pX with

$$\frac{1}{2}\arg\mu = \text{ angle of elevation of the minor axis}$$
$$\frac{1+|\mu|}{1-|\mu|} = \text{ ratio of the major to minor axis}$$

Lecture 17.

All surfaces are to be smooth, oriented, connected and without boundary. All diffeomorphisms are assumed orientation-preserving.

(1) Let X and Y be surfaces and $\varphi : X \to Y$ be a diffeomorphism. Given a conformal structure $\sigma = [g]$ on Y, the *pull-back* $\varphi^*\sigma$ is defined as $[\varphi^* g]$. It is easy to check that the definition is independent of the representative g of σ.

Now assume X and Y are Riemann surfaces. Express φ locally as $w = w(z)$, where z and w are local coordinates on X and Y, and let $\sigma = [\,|dw + \mu(w)\,d\overline{w}|\,]$. Then

$$\begin{aligned}\varphi^*\sigma &= [\,|w_z\,dz + w_{\overline{z}}\,d\overline{z} + \mu(w(z))\,(\overline{w}_z\,dz + \overline{w}_{\overline{z}}\,d\overline{z})|\,] \\ &= [\,|(w_z + \mu(w(z))\,\overline{w_{\overline{z}}})\,dz + (w_{\overline{z}} + \mu(w(z))\,\overline{w_z})\,d\overline{z}|\,] \\ &= \left[\,\left|dz + \frac{w_{\overline{z}} + \mu(w(z))\,\overline{w_z}}{w_z + \mu(w(z))\,\overline{w_{\overline{z}}}}\,d\overline{z}\right|\,\right],\end{aligned}$$

where we have used the fact that

$$\overline{w}_z = \overline{w_{\overline{z}}} \qquad \text{and} \qquad \overline{w}_{\overline{z}} = \overline{w_z}.$$

This shows in particular that $\varphi : X \to Y$ is a biholomorphism if and only if $\varphi^*\sigma_Y = \sigma_X$. It also suggests the following pull-back operation on Beltrami differentials:

$$\varphi^*\left(\mu(w)\,\frac{d\overline{w}}{dw}\right) = \frac{w_{\overline{z}} + \mu(w(z))\,\overline{w_z}}{w_z + \mu(w(z))\,\overline{w_{\overline{z}}}}\,\frac{d\overline{z}}{dz} = \frac{\overline{w_z}}{w_z}\,\frac{\mu(w(z)) + \frac{w_{\overline{z}}}{\overline{w_z}}}{1 + \frac{\overline{w_{\overline{z}}}}{w_z}\,\mu(w(z))}\,\frac{d\overline{z}}{dz}.$$

Thus, at the level of Beltrami coefficients, the pull-back operator acts fiberwise as

$$\varphi^*(\mu) = \lambda\left(\frac{\mu+\alpha}{1+\overline{\alpha}\,\mu}\right) \quad \text{where } \lambda = \frac{\overline{w_z}}{w_z} \quad \text{and} \quad \alpha = \frac{w_{\overline{z}}}{\overline{w_z}}.$$

Note that $|\lambda| = 1$ and $|\alpha| < 1$, where the latter holds since φ is orientation-preserving and hence has positive Jacobian $|w_z|^2 - |w_{\overline{z}}|^2$. It follows that the pull-back operator acts on Beltrami coefficients fiberwise by an automorphism of the unit disk. Note that when φ is a biholomorphism, $\alpha = 0$ and the automorphism reduces to the (linear) rotation $\mu \mapsto \lambda\mu$.

(2) The Integrability Question: "Given a conformal structure σ on a surface X, does there exist a complex structure on X with respect to which $\sigma_X = \sigma$?" If such complex structure exists, we say that it is *compatible* with σ and we call σ *integrable*.

An equivalent question is whether there is a Riemann surface Y and a diffeomorphism $\varphi : X \to Y$ such that $\varphi^*\sigma_Y = \sigma$. Any such φ is said to *rectify* σ. To see the equivalence, note that if $\varphi : X \to Y$ rectifies σ, it pulls back the complex structure of Y to one on X which is compatible with σ. Conversely, if there is a complex structure on X compatible with σ, then the map id : $X \to X$ rectifies σ.

Suppose $\varphi : X \to Y$ and $\psi : X \to Z$ both rectify σ. Then $(\psi \circ \varphi^{-1})^*\sigma_Z = \sigma_Y$, which means $\psi \circ \varphi^{-1} : Y \to Z$ is a biholomorphism. It follows that the map which rectifies a given conformal structure is unique up to post-composition with a biholomorphism.

(3) A local condition for integrability: Suppose $\sigma = [\,|dz + \mu(z)\,d\overline{z}|\,]$ is a conformal structure on a Riemann surface X. Then, by the above computation for pull-backs, the condition $\varphi^*\sigma_Y = \sigma$ translates into

$$\frac{w_{\overline{z}}}{w_z} = \mu(z),$$

which is called the *Beltrami equation*. Any (diffeomorphic) solution of this equation is called a μ-*conformal* map. Note that when $\mu = 0$ it reduces to the classical Cauchy-Riemann equation $w_{\overline{z}} = 0$.

(4) Theorem (Guass): Suppose μ is a smooth complex-valued function defined in $\mathbb{D}$ which satisfies $|\mu(z)| < 1$ at every $z \in \mathbb{D}$. Then there exists a μ-conformal diffeomorphism $\varphi : \mathbb{D} \xrightarrow{\cong} \varphi(\mathbb{D}) \subset \mathbb{C}$.

Note that either $\varphi(\mathbb{D}) = \mathbb{C}$ or $\varphi(\mathbb{D})$ is biholomorphic to $\mathbb{D}$ by the Riemann Mapping Theorem. Thus, in the above theorem we can find μ-conformal solutions $\mathbb{D} \to \mathbb{D}$ or $\mathbb{D} \to \mathbb{C}$.

(5) Corollary: Let X be a surface with a Riemannian metric g. Then around each point of X we can find *isothermal coordinates* (x, y) in which g has the form $\gamma(x, y)\sqrt{dx^2 + dy^2}$.

(6) Corollary: Every smooth conformal structure on a surface is integrable. In particular, every (oriented, connected, boundary-less) surface admits the structure of a Riemann surface.

(7) Corollary (Differential-Geometric Uniformization Theorem): Every simply-connected surface with a Riemannian metric is conformally diffeomorphic to $\mathbb{D}$, $\mathbb{R}^2$ or $\mathbb{S}^2$.

Here "conformal" should be understood as "angle-preserving."

(8) Problem: Let g be a smooth Riemannian metric on $\mathbb{D}$. Decide whether $(\mathbb{D}, g)$ is conformally diffeomorphic to $\mathbb{D}$ or $\mathbb{R}^2$.

The answer is available in certain cases. For example, if $g = |dz + \mu(z)\,d\overline{z}|$ and $\|\mu\|_\infty < 1$, then $(\mathbb{D}, g) \cong \mathbb{D}$ is always the case (this will be a corollary of the Measurable Riemann Mapping Theorem). As another example, suppose g is

rotationally symmetric so that $|\mu|$ depends only on $r = |z|$. Then $(\mathbb{D}, g) \cong \mathbb{D}$ or $\mathbb{R}^2$ according as

$$\lim_{a \to 1^-} \int_{\frac{1}{2}}^{a} \frac{1 + |\mu|}{1 - |\mu|} \frac{dr}{r}$$

is finite or infinite.

(9) Example: Let $g = |dz + k\, d\overline{z}|$ where $0 \leq k < 1$. Then $(\mathbb{D}, g) \cong \mathbb{D}$. In fact, the affine map $w = z + k\overline{z}$ sending $\mathbb{D}$ to the ellipse

$$E = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{(1+k)^2} + \frac{y^2}{(1-k)^2} < 1 \right\}$$

satisfies the Beltrami equation $w_{\overline{z}} = k\, w_z$. Post-composing w with a biholomorphism $E \to \mathbb{D}$ given by the Riemann Mapping Theorem, we obtain a conformal diffeomorphism from $(\mathbb{D}, g)$ to $\mathbb{D}$.

(10) Example: If $g = |dz + z^2\, d\overline{z}|$, then $(\mathbb{D}, g) \cong \mathbb{C}$. In fact, $w = \frac{z}{1-|z|^2}$ is a conformal diffeomorphism from $(\mathbb{D}, g)$ to $\mathbb{C}$ since

$$\frac{w_{\overline{z}}}{w_z} = \frac{\dfrac{z^2}{\overline{(1 - |z|^2)^2}}}{\dfrac{1}{(1 - |z|^2)^2}} = z^2.$$

Lecture 18.

(1) A conformal structure $[\, |dz + \mu(z)\, d\overline{z}| \,]$ on a Riemann surface, or its associated Beltrami differential $\mu(z)\frac{d\overline{z}}{dz}$, is said to have *bounded dilatation* if

$$\|\mu\|_\infty = \sup_{z \in X} |\mu(z)| < 1.$$

(2) An orientation-preserving diffeomorphism $f : X \to Y$ between Riemann surfaces is called *quasiconformal* if $f^*\sigma_Y$ has bounded dilatation. Locally, this means there exists a $0 \leq k < 1$ such that

$$\sup_{z \in X} \left| \frac{f_{\overline{z}}}{f_z} \right| < k.$$

In this case, we say that f is K-quasiconformal, where $1 \leq K = \frac{1+k}{1-k} < +\infty$. Thus, a 1-quasiconformal diffeomorphism is holomorphic.

(3) In many applications, one is bound to consider conformal structures on Riemann surfaces which are only measurable. The integrability question for such conformal structures still makes sense, but maps which would rectify such structures can no longer be smooth. Easy examples show that measurable conformal structures are not generally integrable. However, with the extra assumption of having bounded dilatation, they are integrable and the maps which rectify them are homeomorphisms

which enjoy some degree of regularity. This leads to the idea of considering quasiconformal homeomorphisms between Riemann surfaces.

(4) Let U, V be open sets in $\mathbb{C}$. An orientation-preserving homeomorphism $f : U \to V$ is called K-*quasiconformal* if

(i) f is absolutely continuous on lines (ACL). This means that the restriction of f to almost every horizontal and vertical segment in U is absolutely continuous.

(ii) $|f_{\overline{z}}| \leq k|f_z|$ almost everywhere in U, where $k = \frac{K-1}{K+1}$.

The quantity $\mu_f = \frac{f_{\overline{z}}}{f_z}$ is called the *complex dilatation* of f.

(5) Here are some properties of quasiconformal homeomorphisms:

- If $f : U \to V$ is quasiconformal, then f is differentiable almost everywhere in U, that is, for almost every $p \in U$,

$$f(p+z) = f(p) + zf_z(p) + \overline{z}f_{\overline{z}}(p) + \varepsilon(z),$$

where $\frac{\varepsilon(z)}{z} \to 0$ as $z \to 0$.

- If $f = u + iv : U \to V$ is quasiconformal, the Jacobian

$$\mathrm{J}_f = u_x v_y - u_y v_x = |f_z|^2 - |f_{\overline{z}}|^2$$

is locally integrable in U, and we have

$$\int_E \mathrm{J}_f \, dx\, dy = \mathrm{area}(f(E))$$

for every compact set $E \subset U$. In particular, f maps sets of area zero to sets of area zero.

- The partial derivatives f_z and $f_{\overline{z}}$ of a quasiconformal map $f : U \to V$ are locally square-integrable in U. In fact, if f if K-quasiconformal and $k = \frac{K-1}{K+1}$, then

$$|f_z|^2 \leq \frac{1}{1-k^2}\,\mathrm{J}_f \qquad \text{and} \qquad |f_{\overline{z}}|^2 \leq \frac{k^2}{1-k^2}\,\mathrm{J}_f.$$

- The partial derivatives f_z and $f_{\overline{z}}$ of a quasiconformal map $f : U \to V$ are the distributional derivatives also, that is,

$$\int_U f_z\,\varphi = -\int_U f\,\varphi_z \qquad \text{and} \qquad \int_U f_{\overline{z}}\,\varphi = -\int_U f\,\varphi_{\overline{z}}$$

for every compactly supported smooth test function $\varphi : U \to \mathbb{C}$.

- The standard Chain-Rule formulas hold for the composition of quasiconformal maps: If $w = f(z)$ and $\zeta = g(w)$ are quasiconformal, so is $\zeta = (g \circ f)(z)$,

and the relations

$$\zeta_z = \zeta_w\, w_z + \zeta_{\overline{w}}\, \overline{w}_z$$
$$\zeta_{\overline{z}} = \zeta_w\, w_{\overline{z}} + \zeta_{\overline{w}}\, \overline{w}_{\overline{z}}$$

hold almost everywhere. Dividing, we obtain

$$\mu_{g\circ f} = \frac{\zeta_w\, w_{\overline{z}} + \zeta_{\overline{w}}\, \overline{w}_{\overline{z}}}{\zeta_w\, w_z + \zeta_{\overline{w}}\, \overline{w}_z} = \frac{w_{\overline{z}} + (\mu_g \circ f)\, \overline{w_z}}{w_z + (\mu_g \circ f)\, \overline{w_{\overline{z}}}} = \frac{\mu_f + (\mu_g \circ f)\, (\frac{\overline{w_z}}{w_z})}{1 + (\mu_g \circ f)\, (\frac{\overline{w_z}}{w_z})\, \overline{\mu_f}}.$$

It follows that

$$\mu_{g\circ f} = T\left((\mu_g \circ f)\, (\frac{\overline{w_z}}{w_z})\right) \quad \text{where} \quad s \overset{T}{\mapsto} \frac{s + \mu_f}{1 + \overline{\mu_f} s} \in \text{Aut}(\mathbb{D})$$

- If f is K-quasiconformal, so is f^{-1}.
- If f is K_1-quasiconformal and g is K_2-quasiconformal, the composition $g \circ f$ is K_1K_2-quasiconformal.
- Weyl's Lemma: A 1-quasiconformal homeomorphism is holomorphic.
- If $f : U \to V$ is K-quasiconformal, then

$$K^{-1}\,\text{mod}(A) \leq \text{mod}(f(A)) \leq K\,\text{mod}(A)$$

for every annulus $A \subset U$. According to Ahlfors, this property is equivalent to being K-quasiconformal.

(6) Example: Let $K \geq 1$ and define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(x+iy) = \begin{cases} x + iKy & \text{if } y \geq 0 \\ x + iy & \text{if } y < 0. \end{cases}$$

Then f is an ACL homeomorphism with

$$f_z(x+iy) = \begin{cases} \frac{1+K}{2} & \text{if } y \geq 0 \\ 1 & \text{if } y < 0 \end{cases} \qquad f_{\overline{z}}(x+iy) = \begin{cases} \frac{1-K}{2} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

so that $|\frac{f_{\overline{z}}}{f_z}| \leq \frac{K-1}{K+1}$. It follows that f is K-quasiconformal.

(7) Example: Let $0 \leq k < 1$ and define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} z + k\overline{z} & \text{if } |z| \leq 1 \\ z + \frac{k}{z} & \text{if } |z| > 1. \end{cases}$$

Then f is an ACL homeomorphism with

$$f_z(z) = \begin{cases} 1 & \text{if } |z| \leq 1 \\ 1 - \frac{k}{z^2} & \text{if } |z| > 1 \end{cases} \qquad f_{\overline{z}}(z) = \begin{cases} k & \text{if } |z| \leq 1 \\ 0 & \text{if } |z| > 1 \end{cases}$$

so that $|\frac{f_{\overline{z}}}{f_z}| \leq k$. It follows that f is K-quasiconformal, with $K = \frac{1+k}{1-k}$.

(8) Example: Let $\xi : [0,1] \to [0,1]$ be continuous and non-decreasing, $\xi(0) = 0$, $\xi(1) = 1$, and $\xi'(x) = 0$ almost everywhere (such a function is often called a *devil's staircase*). Extend ξ to a map $\mathbb{R} \to \mathbb{R}$ by setting $\xi(x+n) = \xi(x) + n$ for $n \in \mathbb{Z}$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(x+iy) = x + i(y + \xi(x)).$$

Then f is a homeomorphism which satisfies $f_{\overline{z}} = 0$ almost everywhere in $\mathbb{C}$. However, f is not holomorphic. This does not contradict Weyl's Lemma since f is not ACL, hence not quasiconformal.

(9) Example: The unit disk $\mathbb{D}$ and the complex plane $\mathbb{C}$ are not quasiconformally homeomorphic: If there were a quasiconformal homeomorphism $f : \mathbb{D} \to \mathbb{C}$, then $A = f(\{z : \frac{1}{2} < |z| < 1\})$ would be an annulus of finite modulus. But A contains the punctured disk $\{z : |z| > r\}$ for all large r, whose modulus is infinite.

Lecture 19.

(1) A homeomorphism $f : X \to Y$ between Riemann surfaces is K-quasiconformal if $w \circ f \circ z^{-1}$ is K-quasiconformal for each pair of local coordinates z on X and w on Y for which this composition makes sense.

(2) Much of the notions we discussed above for diffeomorphisms, and the local computations, remain valid for quasiconformal maps, as they are differentiable almost everywhere. Thus, we can talk about measurable Riemannian metrics and conformal structures on surfaces, measurable Beltrami differentials on Riemann surfaces, and the pull-back of a conformal structure or Beltrami differential under a quasiconformal homeomorphism. In particular, if $\varphi : X \to Y$ is a quasiconformal homeomorphism and $\sigma = [\,|dz + \mu(z)\,d\overline{z}|\,]$ a conformal structure on X, then

$$\varphi \text{ rectifies } \sigma \iff \varphi^*\sigma_Y = \sigma \iff \mu_\varphi = \frac{\varphi_{\overline{z}}}{\varphi_z}\,\frac{d\overline{z}}{dz} = \mu \quad \text{a.e.}$$

(3) Theorem (Local solutions of the Beltrami equation): Let μ be a measurable complex-valued function on the unit disk $\mathbb{D}$ with $\|\mu\|_\infty < 1$. Then there exists a quasiconformal homeomorphism $\varphi : \mathbb{D} \to \mathbb{D}$ which satisfies $\frac{\varphi_{\overline{z}}}{\varphi_z} = \mu$ almost everywhere.

(4) The Measurable Riemann Mapping Theorem (MRMT): Let μ be a measurable Beltrami differential on a Riemann surface X which has bounded dilatation. Then there exists a Riemann surface Y and a quasiconformal homeomorphism $\varphi : X \to Y$ such that $\mu_\varphi = \mu$ almost everywhere. If $\psi : X \to Z$ is another such homeomorphism, the map $\psi \circ \varphi^{-1} : Y \to Z$ is biholomorphic.

(5) MRMT with parameters for $\widehat{\mathbb{C}}$: Let μ be a measurable Beltrami differential on the Riemann sphere $\widehat{\mathbb{C}}$ which has bounded dilatation. Then there exists a unique quasiconformal homeomorphism $\varphi^\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi^\mu(0) = 0$, $\varphi^\mu(1) = 1$, $\varphi^\mu(\infty) = \infty$, and $\mu_{\varphi^\mu} = \mu$ almost everywhere. Moreover, if μ depends continuously, smoothly, or analytically on a parameter, so does the normalized solution φ^μ.

(6) A deformation retraction $\mathrm{QC}(\widehat{\mathbb{C}}) \to \mathrm{Aut}(\widehat{\mathbb{C}})$: Let $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal homeomorphism and $\Phi \in \mathrm{Aut}(\widehat{\mathbb{C}})$ be uniquely determined by the condition $\Phi = \varphi$ on the set $\{0, 1, \infty\}$. Define $\mu_t = t\,\mu_\varphi$ for $t \in [0, 1]$. Let $\varphi_t = \Phi \circ \varphi^{\mu_t}$, where φ^{μ_t} is the normalized solution of the Beltrami equation given by MRMT. Then $t \mapsto \varphi_t$ is continuous and by uniqueness of the solutions, $\varphi_1 = \varphi$ and $\varphi_0 = \Phi$.

(7) Let f and g be rational maps and $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal conjugacy between them so that $\varphi \circ f = g \circ \varphi$. The fact that g is holomorphic implies that the Beltrami differential μ_φ is f-invariant, that is $f^*\mu_\varphi = \mu_\varphi$. Conversely, suppose μ is an f-invariant Beltrami differential with $\|\mu\|_\infty < 1$. Then the branched covering $g = \varphi^\mu \circ f \circ (\varphi^\mu)^{-1}$ is a rational map since it is locally 1-quasiconformal away from the branch points.

Thus, there is a correspondence between f-invariant Beltrami differentials of bounded dilatation and rational maps which are quasiconformally conjugate to f (the correspondence need not be one-to-one).

(8) As a basic dynamical application of the preceding remark, let us show that the quasiconformal conjugacy class of a rational map f is always path-connected. Suppose φ is a quasiconformal conjugacy between f and another rational map g. Consider the family $\mu_t = t\mu_\varphi$ of Beltrami differentials as above and note that

$$f^*\mu_t = f^*(t\mu_\varphi) = tf^*\mu_\varphi = t\mu_\varphi = \mu_t,$$

where we have used the fact that the pull-back operator f^* acts as a rotation about the origin and hence is linear. If $\varphi_t = \Phi \circ \varphi^{\mu_t}$ as before, it follows that the path $t \mapsto g_t = \varphi_t \circ f \circ (\varphi_t)^{-1}$ consists of rational maps quasiconformally conjugate to f connecting $g_0 = \Phi \circ f \circ \Phi^{-1}$ to $g_1 = g$. Joining this path to $t \mapsto \Phi_t \circ f \circ \Phi_t^{-1}$ in which $t \mapsto \Phi_t$ is a path in $\mathrm{Aut}(\widehat{\mathbb{C}})$ connecting id to Φ, we obtain the desired path from f to g.

Lecture 20.

Here are 3 elementary applications of MRMT in holomorphic dynamics.

(1) Invariance of multipliers: Let $f(z) = \lambda z + O(z^2)$ be the germ of a holomorphic map in the plane fixing the origin. The multiplier $\lambda = f'(0)$ is clearly invariant under smooth conjugacies. On the other hand, $z \mapsto 2z$ is topologically (even quasiconformally) conjugate to $z \mapsto 3z$.

A remarkable theorem of Naishul asserts that when the origin is an indifferent fixed point, the multiplier λ is invariant under topological conjugacies. Here we prove a weaker version of this result by using MRMT.

Theorem: Let $f(z) = \lambda z + O(z^2)$ and $g(z) = \nu z + O(z^2)$ be quasiconformally conjugate near 0. If $|\lambda| = 1$, then $\lambda = \nu$.

Proof. Let φ be a quasiconformal homeomorphism defined near 0 which satisfies $\varphi(0) = 0$ and $\varphi \circ f = g \circ \varphi$. Consider the Beltrami differential $\mu = \mu_\varphi$ defined near the origin, which is clearly f-invariant. Let $\delta, \varepsilon > 0$ be sufficiently small and define, for $t \in \mathbb{D}(0, 1+\varepsilon)$, the Beltrami differentials

$$\mu_t(z) = \begin{cases} t\,\mu(z) & \text{if } |z| < \delta \\ 0 & \text{otherwise} \end{cases}$$

Since $f^*\mu = \mu$ and f^* is linear, it follows that $f^*\mu_t = \mu_t$ near 0. Let $\varphi_t = \varphi^{\mu_t} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the normalized solution of the Beltrami equation given by MRMT. Then $g_t = \varphi_t \circ f \circ \varphi_t^{-1}$ is a 1-quasiconformal homeomorphism near the origin, hence holomorphic there. Moreover, $t \mapsto g_t(z)$ is holomorphic for each fixed z sufficiently close to 0. Writing $g_t(z) = \lambda_t\, z + O(z^2)$, it follows that $t \mapsto \lambda_t$ is holomorphic. But g_t is conjugate to f whose fixed point at $z = 0$ is indifferent, so $|\lambda_t| = 1$ for all $t \in \mathbb{D}(0, 1+\varepsilon)$, implying $t \mapsto \lambda_t$ is constant. Now $\varphi_0 = \mathrm{id}$ so $g_0 = f$ so $\lambda_0 = \lambda$. Similarly, $\varphi_1 \circ \varphi^{-1}$ is conformal, so g_1 is holomorphically conjugate to g, so $\lambda_1 = \nu$. We conclude that $\lambda = \nu$.

(2) Linearization of hyperbolic germs: A holomorphic germ $f(z) = \lambda z + O(z^2)$ is called *hyperbolic* if $|\lambda| \neq 0, 1$. A classical theorem of Koenigs asserts that every hyperbolic germ is holomorphically linearizable. The classical proof, for $|\lambda| < 1$, consists of showing that the sequence $\{\lambda^{-n} f^{\circ n}(z)\}_{n \geq 1}$ converges uniformly in a neighborhood of the origin to a holomorphic map Φ. It is then clear that $\Phi'(0) = 1$ and $\Phi(f(z)) = \lambda \Phi(z)$. Here we give a proof of this result by applying MRMT.

Theorem (Koenigs): If $f(z) = \lambda z + O(z^2)$ is a hyperbolic germ, there exists a holomorphic change of coordinate $z \mapsto \Phi(z)$ defined near the origin, with $\Phi(0) = 0$, such that $\Phi(f(z)) = \lambda \Phi(z)$.

Proof. Without losing generality, assume $|\lambda| < 1$ (otherwise consider the local inverse of f). Choose a disk $U = \mathbb{D}(0, \varepsilon)$ small enough so that $f(U)$ is compactly contained in U. It then follows by an induction that $f^{\circ n}(U)$ is compactly contained in $f^{\circ n-1}(U)$ for all $n \geq 1$, and that $f^{\circ n}(z) \to 0$ for every $z \in U$. Let L denote the linear contraction $z \mapsto \frac{1}{2}\, z$. Take a smooth diffeomorphism $\psi : A = \{z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 1\} \to \overline{U \smallsetminus f(U)}$ subject only to the condition $\psi(L(z)) = f(\psi(z))$ whenever $|z| = 1$. Extend ψ to a homeomorphism $\overline{\mathbb{D}} \to \overline{U}$ by defining $\psi(L^{\circ n}(z)) =$

$f^{\circ n}(\psi(z))$ for all $n \geq 1$ and all $z \in A$. Then ψ is quasiconformal and satisfies

$$\psi(L(z)) = f(\psi(z)) \qquad \text{for all } z \in \mathbb{D}.$$

Now consider the Beltrami differential $\mu = \mu_\psi$ on $\mathbb{D}$. Extend μ to the entire plane by taking pull-backs under L. The resulting Beltrami differential (still denote by μ) is easily seen to be L-invariant and with bounded dilatation. If $\varphi = \varphi^\mu$ is the normalized solution of the Beltrami equation given by MRMT, it follows that the conjugate homeomorphism $g = \varphi \circ L \circ \varphi^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic. Since $g(0) = 0$, we must have $g(z) = \nu z$ for some $\nu \in \mathbb{C}^*$.

Set $\Phi = \varphi \circ \psi^{-1}$. Then Φ is a 1-quasiconformal homeomorphism defined in a neighborhood of the fixed point 0. By Weyl's Lemma, Φ is holomorphic. Moreover, Φ conjugates f to g near 0, so $\nu = g'(0) = f'(0) = \lambda$.

(3) Construction of Herman rings by surgery: Suppose f is a rational map of degree $d \geq 2$ with a fixed Siegel disk Δ of rotation number θ. Take another rational map g of degree $d' \geq 2$ with a fixed Siegel disk Δ' of rotation number $-\theta$. Following Shishikura, we will construct a rational map F, of degree $d + d' - 1$, with a Herman ring of rotation number θ. The idea is to cut out invariant disks from Δ and Δ' and paste the resulting sphere-with-holes along the boundary to obtain a sphere. There is an obvious action on this sphere coming from the action of f and g on the pieces. We apply MRMT to realize this action as a rational map.

More precisely, let $\phi : \Delta \xrightarrow{\cong} \mathbb{D}(0,2)$ and $\psi : \Delta' \xrightarrow{\cong} \mathbb{D}(0,2)$ be conformal isomorphisms which satisfy

$$\phi(f(z)) = e^{2\pi i\theta}\phi(z) \quad \text{and} \quad \psi(g(z)) = e^{-2\pi i\theta}\psi(z).$$

Let

$$\gamma = \{z \in \Delta : |\phi(z)| = 1\} \quad \text{and} \quad \gamma' = \{z \in \Delta' : |\psi(z)| = 1\}.$$

The mapping $h : \gamma \to \gamma'$ defined by $h(z) = \psi^{-1}(\overline{\phi(z)})$ is a smooth orientation-reversing diffeomorphism which satisfies $h(f(z)) = g(h(z))$ for all $z \in \gamma$. Extend h to a quasiconformal homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with the following properties:

- h maps int(γ) to ext(γ') and ext(γ) to int(γ'). (Here "int" refers to the complementary component of the Jordan curve which contains the center of the Siegel disk and "ext" refers to the other component.)
- h is conformal in a neighborhood of $\widehat{\mathbb{C}} \setminus (\Delta \cap h^{-1}(\Delta'))$.

Define

$$\widetilde{F}(z) = \begin{cases} f(z) & \text{if } z \in \gamma \cup \text{ext}(\gamma) \\ (h^{-1} \circ g \circ h)(z) & \text{if } z \in \text{int}(\gamma) \end{cases}$$

It is easy to check that $\widetilde{F}$ is a degree $d+d'-1$ branched covering of the sphere which is locally quasiconformal away from its branch points. Moreover, $A = \Delta \cap h^{-1}(\Delta')$

is a "topological Herman ring" of rotation number θ for $\widetilde{F}$, and F is holomorphic in a neighborhood of $\widehat{\mathbb{C}} \smallsetminus F^{-1}(A)$.

To conjugate $\widetilde{F}$ to a rational map, define a Beltrami differential μ on $\widehat{\mathbb{C}}$ as follows. First define μ on A by

$$\mu = \begin{cases} \mu_0 & \text{on } A \cap \text{ext}(\gamma) \\ h^*\mu_0 & \text{on } A \cap \text{int}(\gamma) \end{cases}$$

(where μ_0 is the zero Beltrami differential corresponding to the standard conformal structure of the sphere). Clearly, $\widetilde{F} : A \to A$ preserves μ. Extend μ to the union $\bigcup_{n\geq 1} \widetilde{F}^{-n}(A)$ by pulling back via the appropriate iterate of $\widetilde{F}$. Note that only the first pull-back to $\widetilde{F}^{-1}(A) \smallsetminus A$ can possibly increase the dilatation of μ; all further pull-backs are taken by iterates of $\widetilde{F}$ which are holomorphic and so do not change the dilatation. On the complement of this union, set $\mu = \mu_0$. The Beltrami differential μ defined this way is clearly $\widetilde{F}$-invariant and has bounded dilatation. It follows that $F = \varphi^\mu \circ \widetilde{F} \circ (\varphi^\mu)^{-1}$ is a rational map with a Herman ring $\varphi^\mu(A)$ of rotation number θ.

Lecture 21.

We present a simplified version of Sullivan's proof of Fatou's no wandering domain conjecture, following N. Baker and C. McMullen.

(1) Theorem (Sullivan): Let $f \in \text{Rat}_d$ with $d \geq 2$. Then every Fatou component U of f is eventually periodic, that is, there exist $n > m > 0$ such that $f^{\circ n}(U) = f^{\circ m}(U)$.

The idea of the proof is as follows: Assuming there exists a *wandering* Fatou component U (or simply a *wandering domain*), we change the conformal structure of the sphere along the grand orbit of U to find an infinite-dimensional family of rational maps of degree d, all quasiconformally conjugate to f. This is a contradiction since the space Rat_d of rational maps of degree d, as a Zariski open subset of $\mathbb{CP}^{2d+1}$, is finite-dimensional. The eventual periodicity statement for entire maps is false. For example, the map $z \mapsto z + \sin(2\pi z)$ has wandering domains.

(2) Lemma (Baker): If U is a wandering domain for a rational map f, then $f^{\circ n}(U)$ is simply-connected for all large n.

Proof. Let $U_n = f^{\circ n}(U)$. Replacing U by U_k for some large k if necessary, we may assume that no U_n contains a critical point of f, so that $f^{\circ n} : U \to U_n$ is a covering map for all n. We can also arrange that $\infty \in U$. Since the U_n are disjoint subsets of $\mathbb{C} \smallsetminus U$ for $n \geq 1$, we have $\text{area}(U_n) \to 0$. But $\{f^{\circ n}|_U\}$ is a normal family, so every convergent subsequence of this sequence must be a constant function. In particular, $\text{diam}(f^{\circ n}(K)) \to 0$ for every compact set $K \subset U$. Take any loop $\gamma \subset U$ and set $\gamma_n = f^{\circ n}(\gamma) \subset U_n$. By the above argument $\text{diam}(\gamma_n) \to 0$. If B_n is the union of the

bounded components of $\mathbb{C} \smallsetminus \gamma_n$, it follows that $\operatorname{diam}(B_n) \to 0$ also. Since $f(B_n)$ is open, $\partial f(B_n) \subset \gamma_{n+1}$, and $\operatorname{diam} f(B_n) \to 0$, we must have $f(B_n) \subset \overline{B_{n+1}}$ for large n. In particular, the iterated images of B_n are subsets of $\mathbb{C} \smallsetminus U$ for large n. Montel's theorem then implies $B_n \subset F(f)$, which gives $B_n \subset U_n$. Thus γ_n is null-homotopic in U_n for large n. Since $f^{\circ n} : U \to U_n$ is a covering map, we can lift this homotopy to U, which proves U is simply connected.

(3) Let a rational map f have a wandering domain U. In view of the above lemma, we can assume that $U_n = f^{\circ n}(U)$ is simply-connected and $f : U_n \to U_{n+1}$ is a conformal isomorphism for all $n \geq 0$. Given an L^∞ Beltrami differential μ defined on U, we can construct an f-invariant L^∞ Beltrami differential on $\widehat{\mathbb{C}}$ as follows. Use the forward and backward iterates of f to spread μ along the grand orbit

$$GO(U) = \{z \in \widehat{\mathbb{C}} : f^{\circ n}(z) \in U_m \text{ for some } n, m \geq 0\}.$$

On the complement $\widehat{\mathbb{C}} \smallsetminus GO(U)$, set $\mu = \mu_0$. The resulting Beltrami differential is defined almost everywhere on $\widehat{\mathbb{C}}$, it satisfies $f^*\mu = \mu$ by the way it is defined, and $\|\mu\|_\infty < \infty$ since spreading $\mu|_U$ along $GO(U)$ by the iterates of the holomorphic map f does not change the dilatation. Now consider the deformation $\mu_t = t\mu$ for $|t| < \varepsilon$, where $\varepsilon > 0$ is small enough to guarantee $\|\mu_t\|_\infty < 1$ if $|t| < \varepsilon$. Note that since f is holomorphic, f^* acts as a linear rotation, so $f^*\mu_t = \mu_t$. Let $\varphi_t = \varphi^{\mu_t} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the normalized solution of the Beltrami equation $(\varphi_t)_{\bar{z}} = \mu_t (\varphi_t)_z$ which fixes $0, 1, \infty$. Then $f_t = \varphi_t \circ f \circ \varphi_t^{-1}$ is a rational map of degree d, and $t \mapsto f_t$ is holomorphic, with $f_0 = f$. The infinitesimal variation

$$w(z) = \frac{d}{dt}\Big|_{t=0} f_t(z)$$

defines a holomorphic vector field whose value at z lies in the tangent space $T_{f(z)}\widehat{\mathbb{C}}$. In other words, w is a holomorphic section of the pull-back bundle $f^*(T\widehat{\mathbb{C}})$ which in turn can be identified with a tangent vector in $T_f\mathrm{Rat}_d$. This is the so-called *infinitesimal deformation* of f induced by μ. We say that μ induces a *trivial* deformation if $w = 0$.

Another way of describing w is as follows: First consider the unique quasiconformal vector field solution to the equation $\overline{\partial} v = \mu$ which vanishes at $0, 1, \infty$. This is precisely the infinitesimal variation $\frac{d}{dt}|_{t=0}\, \varphi_t(z)$ of the normalized solution of the Beltrami equation. It is not hard to check that $w = \delta_f v$, where

$$\delta_f v(z) = v(f(z)) - f'(z)v(z)$$

measures the deviation of v from being f-invariant. Note in particular that $\delta_f v$ is holomorphic even though v is only quasiconformal, and that $w = \delta_f v$ depends linearly on μ, a fact that is not immediately clear from the first description of w. It follows that μ induces a trivial deformation if and only if v is f-invariant.

It is easy to see that the triviality condition $\delta_f v = 0$ forces v to vanish on the Julia set $J(f)$. In fact, let $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$ be a repelling cycle of f with multiplier λ. Then the condition $\delta_f v = 0$ implies $v(z_{j+1}) = f'(z_j)v(z_j)$ for all $j = 0, \ldots, n-1$, so that

$$\prod_{j=0}^{n-1} v(z_j) = \lambda \cdot \prod_{j=0}^{n-1} v(z_j).$$

Since $|\lambda| > 1$, it follows that $v(z_j) = 0$ for some, hence for all j. Now $J(f)$ is the closure of such cycles and v is continuous, so $v(z) = 0$ for all $z \in J(f)$.

(4) The above construction gives well-defined linear maps

$$B(U) \overset{i}{\hookrightarrow} B(\widehat{\mathbb{C}}, f) \overset{D}{\longrightarrow} T_f \mathrm{Rat}_d$$

Here $B(U)$ is the space of L^∞ Beltrami differentials in U, $B(\widehat{\mathbb{C}}, f)$ is the space of f-invariant L^∞ Beltrami differentials on $\widehat{\mathbb{C}}$, and D is the linear operator $D\mu = w = \delta_f v$ constructed above.

Lemma: $B(U)$ contains an infinite-dimensional subspace $N(U)$ of compactly supported Beltrami differentials with the following property: If $\mu \in N(U)$ satisfies $\mu = \overline{\partial} v$ for some quasiconformal vector field v with $v|_{\partial U} = 0$, then $\mu = 0$.

Assuming this lemma for a moment, let us see how this implies the theorem. Consider the above subspace $N(U)$ for a simply-connected wandering domain U and restrict the above diagram to this subspace. If $D(\mu) = 0$ for some $\mu \in N(U)$, or in other words if μ induces a trivial deformation, that means the normalized solution v to $\overline{\partial} v = \mu$ is f-invariant. Hence $v = 0$ on $J(f)$ and in particular on the boundary of U. By the property of $N(U)$, $\mu = 0$. This means that the infinite-dimensional subspace $N(U)$ injects into $T_f \mathrm{Rat}_d$ whose dimension is $2d + 1$. The contradiction shows that no wandering domain can exist.

(5) It remains to prove the above Lemma. Let us first consider the corresponding problem for the unit disk $\mathbb{D}$. Let $\widehat{N}(\mathbb{D}) \subset B(\mathbb{D})$ be the linear span of the Beltrami differentials $\mu_k(z) = \overline{z}^k \frac{d\overline{z}}{dz}$ for $k \geq 0$. The vector field

$$V_k(z) = \begin{cases} \dfrac{1}{k+1}\, \overline{z}^{k+1} \dfrac{\partial}{\partial z} & |z| < 1 \\ \dfrac{1}{k+1}\, z^{-(k+1)} \dfrac{\partial}{\partial z} & |z| \geq 1 \end{cases}$$

solves the equation $\overline{\partial} V_k = \mu_k$ on $\mathbb{D}$. Let $\mu = \overline{\partial} v \in \widehat{N}(\mathbb{D})$ and $v|_{\partial \mathbb{D}} = 0$, and take the appropriate linear combination V of the V_k which solves $\overline{\partial} V = \mu$. Then $V - v$ is holomorphic in $\mathbb{D}$ and coincides with V on the boundary $\partial \mathbb{D}$. This is impossible if $V|_{\partial \mathbb{D}}$ has any negative power of z in it. Hence $\mu = 0$. To get the compact support condition, let $N(\mathbb{D}) \subset B(U)$ consist of all Beltrami differentials which coincide

with an element of $\widehat{N}(\mathbb{D})$ on the disk $|z| < \frac{1}{2}$ and are zero on $\frac{1}{2} \le |z| < 1$. If $\mu = \overline{\partial} v \in N(\mathbb{D})$ and $v|_{\partial\mathbb{D}} = 0$, then v has to be zero on the annulus $\frac{1}{2} < |z| < 1$ since it is holomorphic there. In particular, it is zero on $|z| = \frac{1}{2}$. Now the same argument applied to the disk $|z| < \frac{1}{2}$ shows $\mu = 0$.

For the general case, consider a conformal isomorphism $\psi : \mathbb{D} \xrightarrow{\cong} U$ with the inverse $\phi = \psi^{-1}$ and define $N(U) = \phi^*(N(\mathbb{D}))$. Let $v = v(z)\frac{\partial}{\partial z}$ be a quasiconformal vector field such that $\mu = \overline{\partial} v \in N(U)$ and $v|_{\partial U} = 0$. Then $\phi_*(v) = v(\psi(z))/\psi'(z)\,\frac{\partial}{\partial z}$ is a vector field on $\mathbb{D}$ which is holomorphic near the boundary $\partial\mathbb{D}$ and $v(\psi(z)) \to 0$ as $|z| \to 1$. By the reflection principle, $v(\psi(z))$ is identically zero near the boundary of $\mathbb{D}$. Since $\psi^*\mu = \overline{\partial}\phi_*(v) \in N(\mathbb{D})$, we must have $\psi^*\mu = 0$, which implies $\mu = 0$.

(6) Sullivan's original argument had to deal with two essential difficulties: (i) the possibility of U being non simply-connected, perhaps of infinite topological type; (ii) the possible complications near the boundary of U, for example when ∂U is not locally-connected. He addressed the former by using a direct limit argument, and the latter by using Carathéodory's theory of "prime ends." Both of these difficulties are surprisingly bypassed in the present proof.

Lecture 22.

(1) Let $A \subset \widehat{\mathbb{C}}$ be a set with at least 4 points and T be a connected complex manifold with base point t_0. A *holomorphic motion of A over (T, t_0)* is a map $\varphi : T \times A \to \widehat{\mathbb{C}}$ such that

(i) $z \mapsto \varphi(t, z)$ is injective for each $t \in T$.

(ii) $t \mapsto \varphi(t, z)$ is holomorphic for each $z \in A$.

(iii) $\varphi(t_0, z) = z$ for every $z \in A$.

In other words, $\{\varphi_t(\cdot) = \varphi(t, \cdot)\}_{t\in T}$ is a holomorphic family of injections of A into $\widehat{\mathbb{C}}$, with $\varphi_{t_0} = \mathrm{id}_A$.

(2) Remarks:

- There is no assumption on the joint continuity of φ in (t, z), or even continuity in z for fixed t. They follow automatically from the λ-Lemma to be discussed below.
- For our purposes, we usually take $(T, t_0) = (\mathbb{D}, 0)$ and call φ a holomorphic motion over $\mathbb{D}$.
- We can always assume that the motion is *normalized* in the sense that $0, 1, \infty$ belong to A and they remain fixed under the motion. To see this, take distinct

points z_1, z_2, z_3 in A and let $\alpha, \beta_t \in \text{Aut}(\widehat{\mathbb{C}})$ be determined by

$$\alpha(0) = z_1 \quad \alpha(1) = z_2 \quad \alpha(\infty) = z_3$$

and

$$\beta_t(\varphi_t(z_1)) = 0 \quad \beta_t(\varphi_t(z_2)) = 1 \quad \beta_t(\varphi_t(z_3)) = \infty.$$

Then $\psi_t = \beta_t \circ \varphi_t \circ \alpha$ is a normalized holomorphic motion of $\alpha^{-1}(A)$.

(3) Examples:

- Let $A = \{0, 1, \infty, a\}$ and $\pi : \mathbb{D} \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ be the holomorphic universal covering map which satisfies $\pi(0) = a$. Then $\{\varphi_t\}_{t\in\mathbb{D}}$ defined by

$$\varphi_t(0) = 0 \quad \varphi_t(1) = 1 \quad \varphi_t(\infty) = \infty \quad \varphi_t(a) = \pi(t)$$

is a holomorphic motion of A over $\mathbb{D}$.

- Let A be the lattice $\mathbb{Z} \oplus i\mathbb{Z}$ and define $\{\varphi_t\}_{t\in\mathbb{H}}$ by

$$\varphi_t(m + in) = m + tn$$

is a holomorphic motion of A over $(\mathbb{H}, i)$.

- Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal homeomorphism and $\mu = \mu_f$. For $|t| < 1$, let $\varphi_t = \varphi^{t\mu}$ be the normalized solution of the Beltrami equation given by MRMT. Then φ_t is a holomorphic motion of $\widehat{\mathbb{C}}$ over $\mathbb{D}$. Thus, every quasiconformal homeomorphism of the sphere gives rise canonically to a holomorphic motion of the sphere.

- Let $U \subsetneq \mathbb{C}$ be a Jordan domain. Suppose there are conformal isomorphisms $f_t^i : U \to U_t^i$ $(i = 0, 1)$ depending holomorphically on a parameter $t \in \mathbb{D}$ such that $\overline{U_t^i} \subset U$ and $U_t^0 \cap U_t^1 = \emptyset$. For every finite word $i_1 \cdots i_n$ of 0's and 1's, let

$$U_t^{i_1\cdots i_n} = f_t^{i_n} \circ \cdots \circ f_t^{i_1}(U)$$

and define the Cantor sets

$$K_t = \bigcap_{n\geq 1} \bigcup U_t^{i_1\cdots i_n}.$$

Then the K_t determine a holomorphic motion of the base Cantor set K_0 over $\mathbb{D}$. To see this, take a $z \in K_0$ and suppose that it is represented by the infinite word $i_1 i_2 i_3 \ldots$ so that

$$z = U_0^{i_1} \cap U_0^{i_1 i_2} \cap U_0^{i_1 i_2 i_3} \cap \cdots$$

Define

$$\varphi(t, z) = U_t^{i_1} \cap U_t^{i_1 i_2} \cap U_t^{i_1 i_2 i_3} \cap \cdots \in K_t.$$

Note that $\varphi(t,z)$ is the locally uniform limit of the sequence of holomorphic functions $\varphi_n(t) = f_t^{i_n} \circ \cdots \circ f_t^{i_1}$, so it depends holomorphically on t. It is now easy to check that $(t,z) \mapsto \varphi(t,z)$ is a holomorphic motion of K_0 over $\mathbb{D}$.

(4) Let $E \subset \widehat{\mathbb{C}}$ be a set with at least 4 points. A homeomorphism $f : E \to f(E) \subset \widehat{\mathbb{C}}$ is called *quasiconformal* if there exists an $M > 0$ such that

$$\mathrm{dist}_{\widehat{\mathbb{C}}\setminus\{0,1,\infty\}}(\chi(f(z_1), f(z_2), f(z_3), f(z_4)), \chi(z_1, z_2, z_3, z_4)) \leq M$$

for all quadruples (z_1, z_2, z_3, z_4) in E. Here $\mathrm{dist}_{\widehat{\mathbb{C}}\setminus\{0,1,\infty\}}$ is the hyperbolic distance in the trice puncture sphere and χ is the cross ratio defined by

$$\chi(z_1, z_2, z_3, z_4) = \frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z_4 - z_2}{z_4 - z_3}.$$

It is not hard to check that this definition of quasiconformality coincides with the standard definition when $E = \widehat{\mathbb{C}}$.

(5) λ-Lemma (Mañe-Sad-Sullivan and Lyubich): A holomorphic motion $\varphi : \mathbb{D} \times A \to \widehat{\mathbb{C}}$ extends uniquely to a holomorphic motion $\Phi : \mathbb{D} \times \overline{A} \to \widehat{\mathbb{C}}$. Moreover, Φ is continuous on $\mathbb{D} \times \overline{A}$ and $\Phi_t : \overline{A} \to \Phi_t(\overline{A})$ is a quasiconformal homeomorphism for each $t \in \mathbb{D}$.

Proof. Without losing generality, assume that the motion is normalized. By Montel's Theorem,

$$\mathcal{F} = \{t \mapsto \varphi(t,z) : z \in A\}$$

is a normal family of holomorphic functions $\mathbb{D} \to \widehat{\mathbb{C}}$, so it has compact closure $\overline{\mathcal{F}}$ in $\mathrm{Hol}(\mathbb{D}, \widehat{\mathbb{C}})$. Moreover, if $f, g \in \overline{\mathcal{F}}$ are distinct, then $f(t) \neq g(t)$ for all $t \in \mathbb{D}$. To see this, take $f_n, g_n \in \mathcal{F}$ such that $f_n \neq g_n$, $f_n \to f$ and $g_n \to g$, and note that $t \mapsto f_n(t) - g_n(t)$ is nowhere vanishing by the injectivity property of holomorphic motions. It follows from Hurwitz Theorem that $t \mapsto f(t) - g(t)$ is nowhere vanishing as well.

For each $t \in \mathbb{D}$ consider the continuous map

$$\pi_t : \overline{\mathcal{F}} \to \widehat{\mathbb{C}} \qquad \pi_t(f) = f(t).$$

By the above observation, π_t is injective. Since $\overline{\mathcal{F}}$ is compact, it follows that π_t is a homeomorphism onto its image, which is easily seen to be the closure of $\varphi_t(A)$. Now

$$\Phi(t,z) = \pi_t \circ \pi_0^{-1}(z) \qquad (t,z) \in \mathbb{D} \times \overline{A},$$

extends φ to a motion of $\overline{A}$.

The definition of the compact-open topology on $\overline{\mathcal{F}}$ shows that for each $r < 1$, the family $\{\pi_t\}_{|t|\leq r}$ is equicontinuous, so the same must be true for the family $\{\Phi_t\}_{|t|\leq r}$. It follows that Φ is continuous on the product $\mathbb{D} \times \overline{A}$.

Finally, choose a quadruple (z_1, z_2, z_3, z_4) in $\overline{A}$ and define a holomorphic map $g : \mathbb{D} \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ by

$$g(t) = \chi(\Phi_t(z_1), \Phi_t(z_2), \Phi_t(z_3), \Phi_t(z_4)).$$

By Schwarz Lemma,

$$\mathrm{dist}_{\widehat{\mathbb{C}} \setminus \{0,1,\infty\}}(g(t), g(0)) \leq \mathrm{dist}_{\mathbb{D}}(t, 0) = \log\left(\frac{1+|t|}{1-|t|}\right),$$

or

$$\mathrm{dist}_{\widehat{\mathbb{C}} \setminus \{0,1,\infty\}}(\chi(\Phi_t(z_1), \Phi_t(z_2), \Phi_t(z_3), \Phi_t(z_4)), \chi(z_1, z_2, z_3, z_4)) \leq \log\left(\frac{1+|t|}{1-|t|}\right),$$

which shows each $\Phi_t : \overline{A} \to \Phi_t(\overline{A})$ is quasiconformal.

(6) The Improved λ-Lemma (Slodkowski): A holomorphic motion $\varphi : \mathbb{D} \times A \to \widehat{\mathbb{C}}$ extends to a holomorphic motion $\Phi : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. The extended motion Φ is continuous on $\mathbb{D} \times \widehat{\mathbb{C}}$ and $\Phi_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is K_t-quasiconformal for each $t \in \mathbb{D}$, where $K_t = \frac{1+|t|}{1-|t|}$.

(7) Remarks:

- It was proved by Sullivan and Thurston that there exists a universal constant $0 < a < 1$ such that every holomorphic motion of A over $\mathbb{D}$ extends to a holomorphic motion of the sphere over the smaller disk $\mathbb{D}(0, a)$. Bers and Royden proved that one can take $a = \frac{1}{3}$. Moreover, their extended motion over $\mathbb{D}(0, \frac{1}{3})$ had the advantage of being canonical in the sense that the Beltrami differential μ_{Φ_t} is harmonic on each component of $\widehat{\mathbb{C}} \setminus \overline{A}$. (A Beltrami differential μ on a hyperbolic Riemann surface X is called *harmonic* if $\mu = \frac{\overline{\phi}}{(\rho_X)^2}$ for some holomorphic quadratic differential ϕ on X.) With this additional property, they proved that the extended motion is unique.
- As Sullivan and Thurston observed, to obtain the improved λ-Lemma, it suffices to prove the following *holomorphic axiom of choice*: Given a finite set A and a point $a \notin A$, every holomorphic motion of A over $\mathbb{D}$ extends to a holomorphic motion of $A \cup \{a\}$ over $\mathbb{D}$.
- In the original version of λ-Lemma, $\mathbb{D}$ can be replaced with an arbitrary connected complex manifold, as essentially the same proof shows. In the Bers-Royden version, $\mathbb{D}$ can be replaced with the unit ball in any complex normed linear space. In the improved λ-Lemma, however, $\mathbb{D}$ cannot be replaced for free; see the next example.

(8) Example (Douady): Let $T = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$, with the base point $t_0 = 2$. Let $A = \{0, 1, 2, \infty\}$ and define a holomorphic motion $\varphi : T \times A \to \widehat{\mathbb{C}}$ by

$$\varphi_t(0) = 0 \quad \varphi_t(1) = 1 \quad \varphi_t(\infty) = \infty \quad \varphi_t(2) = t.$$

This motion is *maximal* in the sense that it cannot be extended to a holomorphic motion of any bigger set over T. To see this, it suffices to show that every holomorphic map $f : T \to T$ has a fixed point. Suppose by way of contradiction that such a fixed point free map exists. By Picard's Great Theorem, none of $0, 1, \infty$ can be an essential singularity for f, so f extends to a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 1$. As $f^{-1}\{0, 1, \infty\} \subset \{0, 1, \infty\}$, f acts bijectively on $\{0, 1, \infty\}$. By the assumption all the $d + 1$ fixed points of f are among $\{0, 1, \infty\}$. If $d = 1$, f is an automorphism which fixes $\{0, 1, \infty\}$ pointwise or fixes one of them and swaps the other two. In either case, it must have a fixed point outside $\{0, 1, \infty\}$, which is a contradiction. If $d > 1$, each fixed point in $\{0, 1, \infty\}$ is a critical point of order $d-1$ and in particular is a simple (i.e., multiplicity 1) fixed point. Since f has $2d-2$ critical points altogether, it follows that f has at most 2 fixed points in $\{0, 1, \infty\}$. Thus $d + 1 \leq 2$, which is again a contradiction.

(9) The analogue of λ-Lemma is certainly false for continuous motions. As an example, let $A = \{\frac{1}{n}\}_{n \geq 1}$ and define the continuous motion $\varphi : \mathbb{R} \times A \to \widehat{\mathbb{C}}$ by $\varphi(t, \frac{1}{n}) = \frac{1}{n} + i\, nt$. Evidently, φ has no continuous extension to $\mathbb{R} \times \overline{A}$.