Math 701 Problem Set 10

due Friday 12/6/2013

$\mathfrak{M}$ and m will denote the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$ and Lebesgue measure, respectively.

Problem 1. For $E \subset \mathbb{R}$ and $\lambda > 0$, set $\lambda E = \{\lambda x : x \in E\}$. Show that if $E \in \mathfrak{M}$, then $\lambda E \in \mathfrak{M}$ and $m(\lambda E) = \lambda m(E)$. (Hint: Verify that the map $t \mapsto \lambda t$ preserves G_δ's, F_σ's, and sets of measure zero.)

Problem 2. Imitate the construction of the middle-thirds Cantor set to show that for every $0 < \alpha < 1$ there is a Cantor set $K \subset [0, 1]$ with $m(K) = \alpha$.

Problem 3. Construct a measurable set $E \in [0, 1]$, with $m(E) = 1/2$, such that both E and $[0, 1] \smallsetminus E$ are dense in $[0, 1]$.

Problem 4. Let $E \subset [0, 1]$ be the set of numbers whose unique non-terminating decimal expansion does not contain the digit 7. Show that $E \in \mathfrak{M}$ and find $m(E)$.

Problem 5. Let $\{E_n\}_{n \geq 1}$ be a sequence in $\mathfrak{M}$ such that $\sum_{n \geq 1} m(E_n) < +\infty$. Show that almost every point of $\mathbb{R}$ belongs to at most finitely many of the E_n. (Hint: Let A be the set of points in $\mathbb{R}$ that belong to infinitely many of the E_n. Verify that

$$A = \bigcap_{k \geq 1} \bigcup_{n \geq k} E_n$$

and conclude that $m(A) = 0$.)

Problem 6. Let $E \subset [0, 1]$ be a measurable set with $m(E) > 0$. Show that the arithmetical difference set

$$E - E = \{x - y : x, y \in E\}$$

contains a neighborhood of the origin. (Hint: Given a small $\varepsilon > 0$, find an open set $U \supset E$ with $m(E) > (1 - \varepsilon)m(U)$. Write U as the disjoint union of open intervals. One of these intervals, say I, must satisfy $m(E \cap I) > (1 - \varepsilon)m(I)$. Show that if $|t| < \delta = (1 - 2\varepsilon)|I|$, then $E \cap I$ meets its translate by t. This will prove $E - E$ contains $(-\delta, \delta)$.)