Math 701 Problem Set 11

due Friday 12/13/2013

$\mathfrak{M}$ and m will denote the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$ and Lebesgue measure, respectively.

Problem 1.

(i) Suppose $f, g : \mathbb{R} \to [-\infty, +\infty]$ are measurable. Show that the sets

$$\{x \in \mathbb{R} : f(x) < g(x)\} \qquad \text{and} \qquad \{x \in \mathbb{R} : f(x) = g(x)\}$$

are measurable.

(ii) Suppose $f_n : \mathbb{R} \to [-\infty, +\infty]$ is a sequence of measurable functions. Show that the set of points at which $\lim_{n\to\infty} f_n$ exists is measurable.

Problem 2. Let $C \subset [0,1]$ be the middle-thirds Cantor set and $f : C \to [0,1]$ be the standard surjection which sends ternary to binary expansions. f extends to a continuous non-decreasing map $f : [0,1] \to [0,1]$ which is constant on every gap of C (this is the so-called "Cantor function" or "Devil's staircase"). Let $g(x) = x + f(x)$. Show that g is continuous and strictly increasing, and $g(C) \subset [0,2]$ has measure 1. Use this to prove that there are measurable subsets of C which map continuously to non-measurable sets (all such measurable sets are therefore non-Borel!).

Problem 3. Suppose $f \in L^1(\mathbb{R})$. Show that there is a constant $C > 0$ such that for all $t > 0$,

$$m\big(\{x \in \mathbb{R} : |f(x)| > t\}\big) < \frac{C}{t}.$$

Problem 4. The ***average value*** of $f \in L^1(\mathbb{R})$ over a set $E \in \mathfrak{M}$ of positive measure is defined by

$$\bar{f}_E = \frac{1}{m(E)} \int_E f \, dm.$$

Prove that $\bar{f}_E \in [a,b]$ for every E if and only if $f(x) \in [a,b]$ for almost every $x \in \mathbb{R}$.

Problem 5. Suppose $f_n : \mathbb{R} \to [0, +\infty]$ are measurable, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, and $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$. If $f_1 \in L^1(\mathbb{R})$, show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm = \int_{\mathbb{R}} f \, dm.$$

Show by an example that the condition $f_1 \in L^1(\mathbb{R})$ cannot be dispensed with.

Problem 6. Suppose $f \in L^1(\mathbb{R})$. Show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $m(E) < \delta$ implies $\int_E |f| \, dm < \varepsilon$. (Hint: Proceed in one of the following ways: (1) Work with the sequence $f_n = \min\{|f|, n\}$ of bounded integrable functions on $\mathbb{R}$ which converges monotonically to $|f|$; or (2) Suppose there is an $\varepsilon > 0$ and sets E_n with $m(E_n) < 1/2^n$ such that $\int_{E_n} |f| \, dm > \varepsilon$ for all n. Look at the integral of $|f|$ over $\bigcup_{k \geq n} E_k$ as $n \to \infty$ to reach a contradiction.)