Math 701 Problem Set 2

due Friday 9/20/2013

Problem 1. Give a direct proof for compactness of $[a, b] \subset \mathbb{R}$ by completing the following sketch: Let $\{U_\alpha\}$ be an open cover for $[a, b]$ and define

$$E = \{x \in [a, b] : [a, x] \text{ is covered by finitely many of the } U_\alpha\}.$$

Show that $\sup E = b \in E$.

Problem 2. Suppose $K \subset U \subset X$, with K compact and U open. Show that $N_\varepsilon(K) \subset U$ if $\varepsilon > 0$ is sufficiently small. Here $N_\varepsilon(K)$ is the ε-neighborhood of K defined by $N_\varepsilon(K) = \bigcup_{p \in K} B(p, \varepsilon)$.

Problem 3. The previous result has a useful generalization in which a single open set is replaced by an arbitrary open cover. Suppose K is compact and $\{U_\alpha\}$ is an open cover for K. Show that there is an $\varepsilon > 0$ such that for every $p \in K$, the ball $B(p, \varepsilon)$ is contained in some U_α. Such ε is called a ***Lebesgue number*** for the cover $\{U_\alpha\}$.

Give an example of an open cover for a non-compact set in $\mathbb{R}$ which does not have a Lebesgue number.

Problem 4.

(i) Prove the following version of Cantor's theorem: *If X is a complete metric space, for every sequence $E_1 \supset E_2 \supset E_3 \supset \cdots$ of non-empty closed sets in X with* $\operatorname{diam} E_n \to 0$*, the intersection $\bigcap_{n \geq 1} E_n$ is a single point.*

(ii) Show by an example that $\bigcap_{n \geq 1} E_n$ may be empty if the condition $\operatorname{diam} E_n \to 0$ is dropped.

(iii) Show that the property in (i) characterizes complete metric spaces. That is, prove that if every decreasing sequence of non-empty closed sets with shrinking diameters nests down to a point, then the ambient space must be complete.

(Hint: For (i), take a point $p_n \in E_n$ and show that $\{p_n\}$ is a Cauchy sequence. For (iii), take a Cauchy sequence $\{p_n\}$ in X and let E_n be the closure of $\{p_n, p_{n+1}, p_{n+2}, \ldots\}$.)

Problem 5. A metric d on a set X is called an ***ultra-metric*** if it satisfies the stronger form of the triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \qquad \text{for all } x, y, z \in X.$$

Suppose d is an ultra-metric on X.

(i) Show that if $d(x, z) \neq d(z, y)$, then $d(x, y) = \max\{d(x, z), d(z, y)\}$.

(ii) Show that a sequence $\{p_n\}$ in (X, d) is Cauchy iff $d(p_n, p_{n+1}) \to 0$ as $n \to \infty$.

Problem 6. Let X be the space of all sequences $P = \{p_n\}_{n\geq 1}$ in a given set. For $P = \{p_n\}$ and $Q = \{q_n\}$ in X, let $n(P, Q)$ be the smallest index $n \in \mathbb{N}$ for which $p_n \neq q_n$ (as usual, we set $n(P, Q) = \infty$ if there is no such n). Define

$$d(P, Q) = \frac{1}{n(P, Q)}.$$

Show that d is an ultra-metric on X and the resulting metric space (X, d) is complete.