Math 701 Problem Set 3

due Friday 9/27/2013

Problem 1. Suppose E and F are closed sets in a metric space, $f : E \to Y$ and $g : F \to Y$ are continuous, and $f = g$ on $E \cap F$. Prove that the map $h : E \cup F \to Y$ defined by $h = f$ on E and $h = g$ on F is continuous. Show by an example that the assumption of E, F being closed is essential. (Hint: Verify that for every closed set $C \subset Y$, $h^{-1}(C)$ is closed in $E \cup F$.)

Problem 2. Let X be a metric space and $E \subset X$ be non-empty. Define the ***distance*** between $x \in X$ and E by

$$\operatorname{dist}(x, E) = \inf_{p \in E} d(x, p).$$

Show that $x \mapsto \operatorname{dist}(x, E)$ is uniformly continuous on X, and vanishes precisely on $\overline{E}$.

Problem 3. Show that balls in $\mathbb{R}^n$ (with the standard metric) are connected. Give an example of a metric space in which there are disconnected balls.

Problem 4. Suppose U is an *open* and connected set in $\mathbb{R}^n$. Prove that U is path-connected. (Hint: Fix a base point $p_0 \in U$ and let E be the set of all points in U that can be joined to p_0 by a path in U. Show that E is both open and closed in U.)

Problem 5.

(i) Let f be a continuous map from a connected metric space X to a metric space Y. Show that the ***graph*** of f defined by

$$\Gamma(f) = \{(x, y) : y = f(x)\} \subset X \times Y$$

is connected. Here on $X \times Y$ you can put any of the several equivalent metrics such as $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

(ii) Show that the subset

$$E = \big\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = \sin(1/x)\big\} \cup \big\{(0, y) : -1 \le y \le 1\big\}$$

of the plane is connected but not path-connected.

(Hint: For (i), note that $\Gamma(f)$ is the image of X under the map $x \mapsto (x, f(x))$. For (ii), observe that E is the closure of a graph.)

Problem 6.

(i) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\sup_{x \in \mathbb{R}} |f'(x)| < +\infty$. Show that f is uniformly continuous on $\mathbb{R}$.

(ii) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, monotonic, and bounded. Show that f is uniformly continuous on $\mathbb{R}$.

(iii) Give an example of a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ which is not uniformly continuous.