Math 701 Problem Set 4

due Friday 10/4/2013

Problem 1. Suppose $-\infty < a < b < +\infty$ and $f : (a,b) \to \mathbb{R}$ is uniformly continuous. Show that $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist.

Problem 2. Let X be a metric space.

(i) Suppose E, F are disjoint closed subsets of X. Show that there are disjoint open sets U, V in X such that $E \subset U$ and $F \subset V$.

(ii) Suppose $p_1, \dots, p_n$ are distinct points in X and $a_1, \dots, a_n$ are real numbers. Show that there exists a continuous function $f : X \to \mathbb{R}$ such that $f(p_i) = a_i$ for every $1 \leq i \leq n$.

(Hint: Use Urysohn's lemma for both parts. In (ii), first argue that there exists a continuous function on X which takes the value 1 at p_i and vanishes at p_j for all $j \neq i$.)

Problem 3. A subset of $\mathbb{R}$ is called a ***G_δ-set*** if it is a countable intersection of open sets.

(i) Show that every closed set in $\mathbb{R}$ is a G_δ-set.

(ii) Show that $\mathbb{R} \smallsetminus \mathbb{Q}$, the set of irrational numbers, is a G_δ-set.

(iii) By contrast, show that $\mathbb{Q}$ is not a G_δ-set.

(Hint: For (i), consider ε-neighborhoods. For (iii), use Baire's theorem.)

Problem 4. Given a function $f : \mathbb{R} \to \mathbb{R}$, let $C(f)$ denote the set of points at which f is continuous.

(i) Prove that

$$C(f) = \bigcap_{n \geq 1} \left\{ x \in \mathbb{R} : \operatorname{diam} f((x-r, x+r)) < \tfrac{1}{n} \text{ for some } r > 0 \right\}.$$

In particular, $C(f)$ is always a G_δ-set.

(ii) Conclude that there is no $f : \mathbb{R} \to \mathbb{R}$ with $C(f) = \mathbb{Q}$.

(iii) By contrast, give an example $f : \mathbb{R} \to \mathbb{R}$ with $C(f) = \mathbb{R} \smallsetminus \mathbb{Q}$.

Problem 5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{n\to\infty} f(nx) = 0$ for every $x \in \mathbb{R}$. Show that $\lim_{x\to\infty} f(x) = 0$. (Hint: For a given $\varepsilon > 0$, apply Baire's theorem to the sets

$$F_k = \{x \in \mathbb{R} : |f(nx)| \leq \varepsilon \text{ for all } n > k\} \qquad k = 1, 2, 3, \dots)$$

Problem 6. Suppose X is a complete metric space and $\mathscr{F}$ is a family of continuous functions $X \to \mathbb{R}$. Assume $\mathscr{F}$ is a pointwise bounded family in the following sense: For each $x \in X$, there is a constant $M_x > 0$ such that $|f(x)| \leq M_x$ for every $f \in \mathscr{F}$. Show that there is a non-empty open set $U \subset X$ and a number $M > 0$ such that $|f(x)| \leq M$ for every $f \in \mathscr{F}$ and every $x \in U$. This is often known as the ***principle of uniform boundedness***. (Hint: Use Baire's theorem.)