Math 701 Problem Set 5

due Friday 10/11/2013

Problem 1. Show that every subset of a separable metric space is separable.

Problem 2. Give an example of a metric space which has a countable perfect subset.

Problem 3. Let X be a complete separable metric space. For a set $E \subset X$, we use the notation E^* for the set of all condensation points of E, i.e., all points $p \in X$ such that $B(p, r) \cap E$ is uncountable for every $r > 0$.

(i) Show that if $P \subset X$ is perfect, then $P = P^*$.

(ii) Show that for any closed set $E \subset X$, the Cantor-Bendixson decomposition $E = P \cup C$ into a perfect set P and an at most countable set C is unique.

(Hint: For (i), use the fact that every perfect set in X is uncountable by Baire's theorem. If $p \in P \smallsetminus P^*$, show that $B(p, r) \cap P$ would have an isolated point for some $r > 0$, which would be an isolated point of P. For (ii), take any decomposition $E = P \cup C$ and use (i) to show that necessarily $P = E^*$ and $C = E \smallsetminus E^*$.)

Problem 4.

(i) Give an explicit example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with the property $|f'(x)| < 1$ for all $x \in \mathbb{R}$ which nevertheless has no fixed point.

(ii) Starting with the number `0` on my calculator screen, I keep pressing the `COS` key over and over again (my calculator is always set in radians!). What happens and why?

Problem 5. Let f be a self-map of a complete metric space X. Suppose for some $k \geq 2$ the k-fold iterate

$$f^{\circ k} = \underbrace{f \circ \cdots \circ f}_{k \text{ times}} : X \to X$$

is a contraction. Show that f has a unique fixed point. (Note that we do not even assume f to be continuous. For example, $f : [0, 2] \to [0, 2]$ which takes the value 2 on $[0, 1)$ and 1 on $[1, 2]$ is discontinuous, but the iterate $f^{\circ 2}$ is the constant function 1, clearly a contraction.)

Problem 6. Suppose X is a complete metric space. Denote the unique fixed point of a contraction $f : X \to X$ by p_f. Show that the assignment $f \mapsto p_f$ is continuous in the following sense: Given f and $\varepsilon > 0$, there is a $\delta > 0$ such that if $g : X \to X$ is any contraction with $\sup_{x \in X} d(f(x), g(x)) < \delta$, then $d(p_f, p_g) < \varepsilon$. (Hint: Estimate $d(f^{\circ n}(x), g^{\circ n}(x))$ inductively on n; the choice $\delta = \varepsilon(1 - \lambda)$ will do, where $\lambda \in [0, 1)$ is the contraction factor of f.)