Math 701 Problem Set 6

due Friday 10/25/2013

The letter C will be reserved for the standard middle-thirds Cantor set in $\mathbb{R}$.

Problem 1. A non-empty compact, perfect, totally disconnected metric space is called a ***Cantor space***. All Cantor spaces are homeomorphic to the standard Cantor set C. Show that if X is a Cantor space and $Y \neq \emptyset$ is a clopen (=closed and open) subset of X, then Y is a Cantor space, so X and Y are homeomorphic.

Problem 2. Show that C is homeomorphic to $C \times C$. (Hint: Check that $C \times C$ is a Cantor space.)

Problem 3. Prove that the arithmetical difference set

$$C - C = \{x - y : x, y \in C\}$$

is the interval $[-1, 1]$. (Hint: For each $t \in [-1, 1]$, use geometry to verify that the line $y = x - t$ intersects $C_n \times C_n$ for every n. Here C_n is the level-n approximation of C consisting of 2^n intervals of length $1/3^n$.)

Problem 4. Show that C is homogeneous in the following sense: Given any pair $p, q \in C$ there exists a homeomorphism $f : C \to C$ such that $f(p) = q$. This shows in particular that the "endpoints" of C (such as 0, 1, $1/3$ or $8/9$) enjoy no topological distinction whatsoever.

Problem 5. Let $\varphi : C \to \Sigma_2$ be the homeomorphism which assigns to each $x \in C$ its unique dyadic address $\varphi(x) = s_1 s_2 s_3 \cdots$ determined by

$$\bigcap_{n \geq 1} I_{s_1 \cdots s_n} = \{x\}.$$

Show that if $0 < \alpha < \log 2 / \log 3$, the map φ is Hölder of exponent α. That is, there exists an $M > 0$ such that

$$d(\varphi(x), \varphi(y)) \leq M|x - y|^{\alpha} \qquad \text{if } x, y \in C.$$

Problem 6. Let $\sigma : \Sigma_2 \to \Sigma_2$ be the ***shift map*** defined by

$$\sigma(s_1 s_2 s_3 \cdots) = s_2 s_3 s_4 \cdots$$

(i) Verify that σ is continuous and 2-to-1.

(ii) How many fixed points does σ have? How many periodic points of period 2? Of period 3? Of a prime period? Of a general period?

(iii) Show that periodic points of σ are dense in Σ_2, that is, given $s \in \Sigma_2$ and $\varepsilon > 0$ there is a $t \in \Sigma_2$ which is fixed by some iterate $\sigma^{\circ k}$, such that $d(s, t) < \varepsilon$.

(iv) Show that there are points $s \in \Sigma_2$ whose orbit $\{s, \sigma(s), \sigma^{\circ 2}(s), \sigma^{\circ 3}(s), \ldots\}$ is dense in Σ_2.

(v) Under the homeomorphism $\varphi : C \to \Sigma_2$, the conjugate map $f = \varphi^{-1} \circ \sigma \circ \varphi : C \to C$ has similar orbit properties as σ. Can you find an explicit formula for f?