Math 701 Problem Set 7

due Friday 11/1/2013

Problem 1. Suppose X is a metric space, $f_n \in \mathscr{C}(X)$, and $f_n \to f$ uniformly on X.

(i) Show that $f_n(p_n) \to f(p)$ in $\mathbb{C}$ whenever $p_n \to p$ in X.

(ii) Show that if each f_n is uniformly continuous on X, so is f.

Problem 2. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}.$$

For what values of x does the series converge? On what intervals does it converge uniformly? Where is f continuous? Is f a bounded function on its domain?

Problem 3. Suppose $f : [a,b] \to \mathbb{R}$ is a C^∞ function such that the sequence $\{f^{(n)}\}$ of its successive derivatives converges uniformly on $[a,b]$. Find the limit function.

Problem 4. Prove that there is no sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(Hint: Use Baire's theorem.)

Problem 5. Fix an integer $d \geq 1$. Let $P_n : \mathbb{R} \to \mathbb{R}$ be a sequence of polynomials of degree $\leq d$. If $P_n(x) \to 0$ for every $x \in [a,b]$, show that $P_n \to 0$ uniformly on $[a,b]$. (Suggestion: First try the easier case of $d = 2$ and $[a,b] = [0,1]$.)

Problem 6.

(i) Let X be a compact metric space. Suppose a sequence of continuous functions $f_n : X \to \mathbb{R}$ converges monotonically increasing to a continuous function $f : X \to \mathbb{R}$ (that is, $f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots$ tend to $f(x)$ for all $x \in X$). Prove that $f_n \to f$ uniformly on X.

(ii) Suppose a sequence of increasing functions $f_n : [a,b] \to \mathbb{R}$ converges pointwise to a continuous function $f : [a,b] \to \mathbb{R}$. Prove that $f_n \to f$ uniformly on $[a,b]$.

(Hint: For (ii), use uniform continuity of f.)