Math 701 Problem Set 8

due Friday 11/15/2013

Problem 1. Suppose $\{P_n\}$ is a sequence of real polynomials such that $P_n \to f$ uniformly on $\mathbb{R}$. What can you say about the limit function f?

Problem 2. Let $\{f_n\}$ and g be Riemann integrable functions on $[0,1]$ such that $|f_n(x)| \leq g(x)$ for all x and n. Define

$$F_n(x) = \int_0^x f_n(t)\,dt.$$

Show that $\{F_n\}$ has a subsequence which converges uniformly on $[0,1]$.

Problem 3. Let $\{f_n\}$ be a sequence of twice differentiable functions on $[a,b]$ such that $f_n(a) = f_n'(a) = 0$ for all n. Assume further that there is an $M > 0$ such that $|f_n''(x)| \leq M$ for all x and n. Show that $\{f_n\}$ has a subsequence which converges uniformly on $[a,b]$. (Hint: Mean value theorem and Arzelà-Ascoli are helpful here.)

Problem 4.

(i) Suppose $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^1 x^n f(x)\,dx = 0$ for all integers $n \geq 0$. Show that $f = 0$ on $[0,1]$.

(ii) Repeat part (i) assuming that $\int_0^1 x^n f(x)\,dx = 0$ for all large enough n.

(Hint: In (i), the integral of f against any polynomial is zero. Use the Weierstrass approximation theorem to show that $\int_0^1 f^2 = 0$. For (ii), well..., isn't it obvious?)

Problem 5. Suppose X is a compact metric space. A subset $I \subset \mathscr{C}(X)$ is called an ***ideal*** if $f, g \in I$ and $h \in \mathscr{C}(X)$ imply $f + g \in I$ and $fh \in I$. An ideal $I \neq \mathscr{C}(X)$ is called ***maximal*** if for any ideal J, the inclusions $I \subset J \subset \mathscr{C}(X)$ imply $J = I$ or $J = \mathscr{C}(X)$ (that is, there is no ideal properly between I and $\mathscr{C}(X)$).

(i) Verify that if $p \in X$, then

$$I_p = \{f \in \mathscr{C}(X) : f(p) = 0\}$$

is a maximal ideal.

(ii) Show that every maximal ideal in $\mathscr{C}(X)$ is of the form I_p for some $p \in X$.

(Hint for (ii): Let I be a maximal ideal. If for every $p \in X$ there is an $f_p \in I$ with $f_p(p) \neq 0$, construct a function in I which does not vanish anywhere, and conclude that $I = \mathscr{C}(X)$, which is a contradiction.)

Problem 6. A special case of the Stone-Weierstrass theorem shows that polynomials $\sum a_{ij} x^i y^j$ in two variables are dense in the space of all continuous functions $[a,b] \times [c,d] \to \mathbb{R}$. Use the Stone-Weierstrass theorem to prove the following analog in a general product space: Suppose X, Y are compact metric spaces and $f : X \times Y \to \mathbb{R}$ is continuous. Then, for every $\varepsilon > 0$ there are continuous functions $f_i : X \to \mathbb{R}$ and $g_i : Y \to \mathbb{R}$

$(1 \leq i \leq n)$ such that

$$\sup_{(x,y)\in X\times Y} \left| f(x, y) - \sum_{i=1}^{n} f_i(x) g_i(y) \right| < \varepsilon.$$

Thus, continuous functions of two variables can be uniformly approximated on compact sets by finite combinations of products of functions of a single variable.