Math 701 Problem Set 9

due Friday 11/22/2013

Problem 1. Suppose $f : [a, b] \to \mathbb{R}$ is increasing, that is, $x \leq y$ implies $f(x) \leq f(y)$. Show that $f \in \mathscr{R}[a, b]$. (Hint: Take a partition P of $[a, b]$ into equal pieces and compute $U(f, P) - L(f, P)$.)

In the next three problems the Lebesgue-Riemann integrability criterion will be useful.

Problem 2. Recall that the ***characteristic function*** of a set $E \subset [a, b]$ is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Under what condition on E is it true that $\chi_E \in \mathscr{R}[a, b]$?

Problem 3.

(i) Suppose $f \in \mathscr{R}[a, b]$ and $\phi : \mathbb{R} \to \mathbb{R}$ is continuous. Show that $\phi \circ f \in \mathscr{R}[a, b]$.

(ii) Show that $f \in \mathscr{R}[a, b]$ implies $f^2 \in \mathscr{R}[a, b]$.

(iii) Show that $f^2 \in \mathscr{R}[a, b]$ does not imply $f \in \mathscr{R}[a, b]$.

(iv) Show that $f^3 \in \mathscr{R}[a, b]$ implies $f \in \mathscr{R}[a, b]$.

Problem 4. Suppose $f \in \mathscr{R}[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$. Show that $\int_a^b f > 0$.

Problem 5. If $f : \mathbb{R} \to \mathbb{R}$ is C^1 and $E \subset \mathbb{R}$ has measure zero, prove that the image $f(E)$ has measure zero. Show by an example that this is no longer true if f is only assumed to be continuous.

Problem 6. Recall that a subset of $\mathbb{R}$ is ***meager*** if it is contained in a countable union of nowhere dense sets. Show that there is a meager set $E \subset \mathbb{R}$ whose complement has measure zero. Thus, in the decomposition $\mathbb{R} = E \cup E^c$, both sets are small but in different meanings. (Hint: For each n, cover $\mathbb{Q}$ by countably many open intervals of total length $< 1/n$.)