Math 702 Problem Set 10

due Friday 5/2/2014

Unless otherwise stated, X is a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The null-space and range of a linear map $T : X \to X$ are denoted by $N(T)$ and $R(T)$.

Problem 1. This exercise proves a family of polarization identities in inner product spaces.

(i) Let $\zeta = e^{2\pi i/n}$, where $n \geq 1$ is an integer. Show that

$$\frac{1}{n}\sum_{k=1}^{n} \zeta^{kj} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } 1 \leq j \leq n-1. \end{cases}$$

(ii) Use (i) to show that for every x, y in an inner product space,

$$\langle x, y \rangle = \frac{1}{n}\sum_{k=1}^{n} \|x + \zeta^k y\|^2 \, \zeta^k$$

provided that $n \geq 3$.

(iii) Prove the following continuous version of the above identity:

$$\langle x, y \rangle = \frac{1}{2\pi}\int_0^{2\pi} \|x + e^{it} y\|^2 \, e^{it} \, dt.$$

Problem 2. Assuming Y is a closed subspace of X, prove the following statements:

(i) $(Y^\perp)^\perp = Y$.

(ii) The natural map $Y^\perp \to X/Y$ given by $z \mapsto z + Y$ is an isometric isomorphism (recall that the norm on X/Y is defined by $\|x + Y\| = \inf_{y \in Y} \|x + y\|$).

(iii) If $x \in X$,

$$\min \left\{ \|x - y\| : y \in Y \right\} = \max \left\{ |\langle x, z \rangle| : z \in Y^\perp, \|z\| = 1 \right\}.$$

Problem 3. Suppose f is a non-zero bounded linear functional on X. Prove the following statements:

(i) $(\ker f)^\perp$ has dimension 1.

(ii) $\ker f = \ker g$ for a bounded linear functional g if and only if $f = \lambda g$ for some scalar λ.

Problem 4. Take advantage of the fact that $L^2[-1,1]$ is a Hilbert space to compute

$$\min_{a,b,c\in\mathbb{C}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx$$

and

$$\max_{g} \left| \int_{-1}^{1} x^3 g(x) \, dx \right|,$$

where $g \in L^2[-1,1]$ is subject to the restrictions

$$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} xg(x)\,dx = \int_{-1}^{1} x^2 g(x)\,dx = 0 \quad \text{and} \quad \int_{-1}^{1} |g(x)|^2\,dx = 1.$$

Problem 5. Let $T : X \to X$ be a bounded linear map.

(i) Show that there is a unique bounded linear map $T^* : X \to X$, called the ***adjoint*** of T, which satisfies

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for every $x, y \in X$.

(ii) Verify the relations

$$\|T^*\| = \|T\| \qquad \text{and} \qquad \|TT^*\| = \|T^*T\| = \|T\|^2.$$

(iii) Show that $N(T^*) = R(T)^\perp$ and $N(T) = R(T^*)^\perp$.

Problem 6.

(i) Let Y be a closed subspace of X and $P : X \to X$ be the orthogonal projection onto Y (so $R(P) = Y$ and $N(P) = Y^\perp$). Show that $P = P^2 = P^*$ and $\|P\| = 1$.

(ii) Let $P : X \to X$ be a bounded linear map such that $P = P^2 = P^*$. Show that $R(P)$ is a closed subspace of X and P is the orthogonal projection onto $R(P)$.

Problem 7. (Bonus) Suppose $T : X \to X$ is linear and

$$\langle T(x), y\rangle = \langle x, T(y)\rangle$$

for every $x, y \in X$. Show that T is bounded (hence $T = T^*$).