Math 702 Problem Set 11

due Friday 5/9/2014

Unless otherwise stated, X is a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$.

Problem 1. Let (X, μ) be a measure space in which there are disjoint measurable sets of finite positive measure. Show that if $p \neq 2$, the L^p-norm on X does not satisfy the parallelogram law

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2.$$

Conclude that the Banach space $L^p(\mu)$ is not a Hilbert space.

Problem 2. Let $\{x_i\}$ be a sequence of orthogonal vectors in X. Show that the following conditions are equivalent:

(i) $\sum_{i=1}^{\infty} \|x_i\|^2$ converges;

(ii) $\sum_{i=1}^{\infty} x_i$ converges in X;

(iii) $\sum_{i=1}^{\infty} \langle x_i, y \rangle$ converges for every $y \in X$.

(Hint: The Pythagorean theorem is helpful here. For (iii) $\Longrightarrow$ (i), apply the uniform boundedness principle to the functionals $f_n(y) = \sum_{i=1}^{n} \langle y, x_i \rangle$.)

Problem 3. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Prove the following statements:

(i) If $\|x_n\| \leq 1$, $\|y_n\| \leq 1$, and $\langle x_n, y_n \rangle \to 1$, then $\|x_n - y_n\| \to 0$.

(ii) If $x_n \overset{\text{w}}{\to} x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

(Hint: For (ii), use Riesz's theorem according to which every element of X^* is of the form $x \mapsto \langle x, y \rangle$ for some $y \in X$.)

Problem 4. Let $x \in X$ and $\hat{x}_\alpha = \langle x, u_\alpha \rangle$ be the Fourier coefficients of x with respect to a given orthonormal basis $\{u_\alpha\}$ for X. Prove the following statements:

(i) The set $\{\alpha : \hat{x}_\alpha \neq 0\}$ is at most countable.

(ii) If $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ is the set in (i),

$$x = \sum_n \hat{x}_{\alpha_n} u_{\alpha_n},$$

Thus, the "Fourier series" of x converges to x in the norm topology of X.

Problem 5. Recall that an isomorphism between two Hilbert spaces is a bijective linear map which is inner product-preserving (equivalently, norm-preserving).

(i) Show that $\ell^2(A)$ is isomorphic to $\ell^2(B)$ if and only if A and B have the same cardinality.

(ii) Show that a Hilbert space is separable if and only if it has an orthonormal basis which is at most countable.

(iii) Conclude that every infinite-dimensional separable Hilbert space is isomorphic to $\ell^2 = \ell^2(\mathbb{N})$.

Problem 6. Recall that $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $L^2(\mathbb{T})$ is the space of all measurable functions on $\mathbb{T}$ (identified with 1-periodic functions on $\mathbb{R}$) such that $\|f\|_2 = (\int_0^1 |f(t)|^2\,dt)^{1/2} < \infty$. Let $0 < \alpha < 1$ be irrational and $f \in L^2(\mathbb{T})$ satisfy $f(t+\alpha) = f(t)$ for a.e. $t \in \mathbb{T}$. Show that f is constant a.e. on $\mathbb{T}$.