Math 702 Problem Set 12

Throughout $\mathfrak{M}$ is a σ-algebra in a set X. As usual, the acronym AC stands for "absolutely continuous."

Problem 1. Suppose μ, λ are complex measures on $\mathfrak{M}$, and $E \in \mathfrak{M}$. Prove the following statements:

(i) $|\mu + \lambda|(E) \leq |\mu|(E) + |\lambda|(E)$.

(ii) $|\mu|(E) = \sup \left\{ \sum_{i=1}^{n} |\mu(E_i)| : E_1, \ldots, E_n \text{ is a finite partition of } E \right\}$.

Problem 2. Consider the relation $\ll$ on the space of finite positive measures on $\mathfrak{M}$.

(i) Prove that $\ll$ is transitive, and if $\mu \ll \lambda$ and $\lambda \ll \nu$, then the chain rule

$$\frac{d\mu}{d\nu} = \frac{d\mu}{d\lambda} \frac{d\lambda}{d\nu}$$

holds ν-a.e. in X.

(ii) If $\mu \ll \lambda$ and $\lambda \ll \mu$ (such pairs of measures are said to be ***equivalent***), how do the Radon-Nikodym derivatives $d\mu/d\lambda$ and $d\lambda/d\mu$ relate?

Problem 3. Let m and μ denote Lebesgue measure and the counting measure on $\mathbb{R}$, respectively.

(i) Show that despite $m \ll \mu$ there is no f for which $dm = f\, d\mu$.

(ii) Show that μ has no Lebesgue decomposition with respect to m.

Why don't these failures contradict the Lebesgue-Radon-Nikodym theorem?

Problem 4. Prove that the Jordan decomposition of a signed measure is minimal: If ν is a signed measure on $\mathfrak{M}$ and if $\nu = \mu - \lambda$ for some finite positive measures μ, λ on $\mathfrak{M}$, then $\mu \geq \nu^+$ and $\lambda \geq \nu^-$. (Hint: Use the Hahn decomposition theorem.)

Problem 5. Suppose $f : [a, b] \to \mathbb{R}$ is AC and $f' \in L^p[a, b]$ for some $1 < p < \infty$. Show that f is Hölder continuous of exponent $1/q$, where $1/p + 1/q = 1$.

Problem 6. Construct a homeomorphism $f : [0, 1] \to [0, 1]$ such that $f'(x) = 0$ for almost every $x \in [0, 1]$. (Hint: You may want to think about an infinite sum of suitably scaled Cantor functions.)

Problem 7. Show that if f, g are AC on $[a, b]$, so is their product fg. Use this to prove the integration by parts formula

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

Problem 8. Recall that a function $f : [0,1] \to \mathbb{R}$ is ***M-Lipschitz*** if $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [0,1]$. Prove that f is M-Lipschitz if and only if there exists a sequence $\{f_n\}$ of continuously differentiable functions defined on $[0,1]$ such that

(i) $|f_n'(x)| \leq M$ for all n and all $x \in [0,1]$, and

(ii) $f_n(x) \to f(x)$ for all $x \in [0,1]$ as $n \to \infty$.