Math 702 Problem Set 2

due Friday 2/21/2014

Unless otherwise stated, $(X, \mathfrak{M}, \mu)$ is a given measure space. A measurable function $f : X \to [0, +\infty]$ is integrable if $\int_X f \, d\mu < +\infty$.

Problem 1. Suppose $f : X \to [0, +\infty]$ is measurable.

(i) If f is integrable, show that $f < +\infty$ almost everywhere on X.

(ii) If $\int_X f \, d\mu = 0$, show that $f = 0$ almost everywhere on X.

Problem 2. Suppose $f : X \to [0, +\infty]$ is integrable. Show that there is a constant $C > 0$ such that for all $t > 0$,

$$\mu\big(\{x \in X : f(x) > t\}\big) < \frac{C}{t}.$$

Is there an example in which the measure on the left tends to 0 like $1/t$ as $t \to +\infty$? Tends to $+\infty$ like $1/t$ as $t \to 0^+$?

Problem 3. *(Borel-Cantelli's lemma revisited)* Recall from the last assignment that if $\{E_n\}$ is a sequence of measurable sets in X such that $\sum \mu(E_n) < +\infty$, then almost every point in X belongs to at most finitely many of the E_n. The recommended proof was purely measure-theoretic with no reference to integration. Use the monotone convergence theorem to give another proof for this lemma. Show by an example that the result no longer holds under the weaker assumption $\lim_{n\to\infty} \mu(E_n) = 0$.

Problem 4. Suppose $f_n : X \to [0, +\infty]$ are measurable, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, and $f_n(x) \to f(x)$ for all $x \in X$. If f_1 is integrable, show that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Show by an example that integrability of f_1 is an essential assumption.

Problem 5. Suppose $f_n : X \to [0, +\infty)$ are measurable and $f_n \to f$ uniformly on X. If $\mu(X) < +\infty$, show that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Show by an example that the condition $\mu(X) < +\infty$ cannot be dispensed with.

Problem 6. *(Change of variable formula)* Let $\varphi : X \to Y$ be a map and equip Y with the push-forward σ-algebra $\varphi_* \mathfrak{M}$ and measure $\varphi_* \mu$. Show that for every measurable function $f : Y \to [0, +\infty]$,

$$\int_Y f \, d(\varphi_* \mu) = \int_X (f \circ \varphi) \, d\mu.$$