Math 702 Problem Set 4

due Friday 3/7/2014

Unless otherwise stated, X is a metric space.

Problem 1. A function $f : X \to [-\infty, +\infty]$ is called ***lower semi-continuous (l.s.c.)*** if $f^{-1}(a, +\infty]$ is open for every $a \in \mathbb{R}$, and ***upper semi-continuous (u.s.c.)*** if $f^{-1}[-\infty, a)$ is open for every $a \in \mathbb{R}$. Prove the following statements:

(i) f is continuous iff it is both l.s.c. and u.s.c.

(ii) The characteristic function χ_E is l.s.c. iff E is open, and is u.s.c. iff E is closed.

(iii) Local formulation of semi-continuity:

$$f \text{ is l.s.c.} \iff \liminf_{x \to p} f(x) \geq f(p) \text{ for every } p \in X$$

$$f \text{ is u.s.c.} \iff \limsup_{x \to p} f(x) \leq f(p) \text{ for every } p \in X.$$

(iv) If f_α is l.s.c. (resp. u.s.c.) for each α, so is $\sup_\alpha f_\alpha$ (resp. $\inf_\alpha f_\alpha$). Here α runs over an arbitrary index set.

(v) If X is compact, a l.s.c. (resp. u.s.c.) function $X \to \mathbb{R}$ reaches its minimum (resp. maximum) at some point of X.

(vi) If X is locally compact and $f : X \to [0, +\infty]$ is l.s.c., then

$$f(x) = \sup \{g(x) : g \in \mathscr{C}_c(X) \text{ and } 0 \leq g \leq f\}.$$

(Hint for (vi): Use Urysohn's lemma.)

Problem 2. Recall that the space $\mathscr{C}[0, 1]$ of continuous complex-valued functions on $[0, 1]$, equipped with the distance

$$\|f - g\|_\infty = \sup_{x \in [0,1]} |f(x) - g(x)|,$$

is a complete separable metric space.

(i) Show that the closed unit ball $\{f \in \mathscr{C}[0, 1] : \|f\|_\infty \leq 1\}$ is not compact.

(ii) Show that every compact subset of $\mathscr{C}[0, 1]$ has empty interior. Conclude that $\mathscr{C}[0, 1]$ is not locally compact.

(iii) Show that $\mathscr{C}[0, 1]$ is not σ-compact.

(Hint: For (i), find a sequence in the closed unit ball with no convergent subsequence. For (iii), use Baire's theorem.)

Problem 3. Describe the compact sets $K \subset \mathbb{R}$ for which there is an $f \in \mathscr{C}_c(\mathbb{R})$ with $\operatorname{supp}(f) = K$.

Problem 4. Show that the sum of two Radon measures is Radon.

Problem 5. Equip $\mathbb{R}$ with the discrete metric (so all subsets are now open) and let $\{r_n\}_{n\geq 1}$ be an enumeration of $\mathbb{Q}$. Is the measure $\mu : 2^{\mathbb{R}} \to [0, +\infty)$ defined by

$$\mu(E) = \sum_{n: r_n \in E} 2^{-n}$$

a Radon measure?

Problem 6. *(Intermediate value property of measures)* Suppose μ is a Radon measure on X with the property that $\mu(\{x\}) = 0$ for every $x \in X$. Let $E \subset X$ be a Borel set with $0 < \mu(E) < +\infty$. Prove that for every $0 < a < \mu(E)$ there is a Borel set $A \subset E$ such that $\mu(A) = a$. (Hint: Every point of X has neighborhoods of arbitrarily small measure.)