Math 702 Problem Set 7

due Friday 3/28/2014

The symbol m_k denotes Lebesgue measure on $\mathbb{R}^k$.

Problem 1. The ***graph*** and ***undergraph*** of a measurable function $f : \mathbb{R} \to [0, +\infty)$ are respectively defined by

$$\Gamma_f = \{(x, y) \in \mathbb{R}^2 : y = f(x)\} \quad \text{and} \quad U_f = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq f(x)\}.$$

(i) Show that Γ_f is measurable and $m_2(\Gamma_f) = 0$.

(ii) Show that U_f is measurable and $m_2(U_f) = \int_{-\infty}^{\infty} f(x)\, dx$.

(Hint: For measurability issues, look at the function $(x, y) \mapsto f(x) - y$. For computations, use Fubini-Tonelli.)

Problem 2. Let $f : (0, +\infty) \to (0, +\infty)$ be an orientation-reversing homeomorphism (so f is continuous and strictly decreasing, with $\lim_{x \to 0^+} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = 0$). Show that

$$\int_0^{\infty} f(x)\, dx = \int_0^{\infty} f^{-1}(x)\, dx.$$

Problem 3. Verify that the function $\sin x/x$ is not Lebesgue integrable on $[0, +\infty)$. On the other hand, use the Fubini-Tonelli theorem and the relation

$$\frac{1}{x} = \int_0^{\infty} e^{-xt}\, dt \qquad (x > 0)$$

to show that

$$\lim_{b \to \infty} \int_0^b \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

This is another illustration of the difference between Lebesgue integral and improper Riemann integral.

Problem 4. Let $\mathbb{S}^{k-1} = \{u \in \mathbb{R}^k : \|u\| = 1\}$ be the unit sphere in $\mathbb{R}^k$. Identify the product $\mathbb{S}^{k-1} \times (0, +\infty)$ with $\mathbb{R}^k \smallsetminus \{0\}$ by the homeomorphism $\phi : (u, r) \mapsto ru$. Define a Borel measure μ on $\mathbb{S}^{k-1}$ by

$$\mu(E) = k\, m_k(\phi(E \times (0, 1))).$$

Show that for any positive Borel function $f : \mathbb{R}^k \to \mathbb{R}$,

$$\int_{\mathbb{R}^k} f\, dm_k = \int_0^{\infty} \int_{\mathbb{S}^k} r^{k-1} f(ru)\, d\mu\, dr.$$

Interpret this formula for $k = 2$.

Problem 5. Recall that the sections of $f : \mathbb{R}^2 \to \mathbb{R}$ are the functions $f_x, f^y : \mathbb{R} \to \mathbb{R}$ defined by

$$f_x(y) = f^y(x) = f(x, y) \qquad \text{for all } x, y \in \mathbb{R}.$$

(i) If f_x is Borel measurable for all x and f^y is continuous for all y, show that f is Borel measurable.

(ii) If f_x is Lebesgue measurable for all $x \in \mathbb{Q}$ and f^y is continuous for a.e. y, show that f is Lebesgue measurable.

Problem 6. The ***Fourier transform*** of $f \in L^1(\mathbb{R})$ is defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx.$$

(i) Show that $\hat{f}$ is a continuous function on $\mathbb{R}$ and $\|\hat{f}\|_\infty \leq \|f\|_1$.

(ii) If $f > 0$ everywhere, show that $|\hat{f}(\omega)| < \hat{f}(0)$ for all $\omega \neq 0$.

(iii) If $h = f * g$ (the convolution), prove that $\hat{h} = \hat{f}\ \hat{g}$.