Math 702 Problem Set 8

due Friday 4/11/2014

In the following problems, (X, μ) is always a measure space,

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p} \qquad (0 < p < \infty),$$

and $L^p = L^p(\mu)$ is the space of all measurable $f : X \to \mathbb{C}$ for which $\|f\|_p < \infty$.

Problem 1. Let $f : X \to \mathbb{C}$ be measurable. Prove the following statements:

(i) If $0 < r < p < s \leq \infty$ and $f \in L^r \cap L^s$, then $f \in L^p$. In other words, the set $\{p \in (0, \infty] : f \in L^p\}$ is connected.

(ii) If $0 < r < s \leq \infty$ and $f \in L^r \cap L^s$, and if f is not a.e. equal to 0, the function $\varphi(p) = \log \|f\|_p^p$ is convex in the interval (r, s).

(Hint: For (i), consider the sets where $|f| < 1$ and $|f| \geq 1$. For (ii), take $x, y \in (r, s)$ and $0 < t < 1$ and use Hölder's inequality for $p = 1/t, q = 1/(1-t)$ to show $\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y)$.)

Problem 2. Suppose $f : X \to \mathbb{C}$ is measurable and $f \in L^r$ for some $0 < r < \infty$. Prove the following statements about the behavior of $\|f\|_p$ for large and small p:

(i) $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ whether or not $\|f\|_\infty$ is finite.

(ii) If $\mu(X) = 1$, then

$$\lim_{p\to 0} \|f\|_p = \exp\left(\int_X \log|f| \, d\mu\right)$$

if we define $\exp(-\infty) = 0$.

(Hint: For (ii), use Jensen's inequality for one direction, and $\log x \leq x - 1$ together with $\lim_{p\to 0}(|f|^p - 1)/p = \log|f|$ for the other direction.)

Problem 3. Suppose $\mu(X) < \infty$, $f \in L^\infty(\mu)$, and $\|f\|_\infty > 0$. Define

$$\alpha_n = \int_X |f|^n \, d\mu, \qquad n = 1, 2, 3, \ldots$$

Show that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

Problem 4. For each $0 < p \leq \infty$, find a function in $L^p(\mathbb{R})$ that does not belong to $L^r(\mathbb{R})$ for any $r \neq p$.

Problem 5. The ***essential range*** of $f \in L^\infty(\mu)$ is the set R_f of points $z \in \mathbb{C}$ for which $\mu(\{x \in X : |f(x) - z| < \varepsilon\})$ is positive for every $\varepsilon > 0$. Show that R_f is compact. What relation can you establish between R_f and $\|f\|_\infty$?

Problem 6. A sequence $f_n : X \to \mathbb{C}$ of measurable functions is said to ***converge in measure*** to a measurable function $f : X \to \mathbb{C}$ if for every $\varepsilon > 0$,

$$\mu\big(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Prove the following statements:

(i) If $\mu(X) < \infty$ and $f_n \to f$ a.e., then $f_n \to f$ in measure.

(ii) If $f_n \to f$ in $L^p(\mu)$ for some $1 \leq p \leq \infty$, then $f_n \to f$ in measure.

(iii) If $f_n \to f$ in measure, some subsequence of f_n converges to f a.e.

Show that the converses of (i) and (ii) do not hold, so convergence in measure is weaker than both a.e. pointwise convergence and L^p convergence. (Hint: For (i), use Egoroff's theorem. For (iii), the Borel-Cantelli lemma is useful.)