Math 702 Problem Set 9

due Friday 4/25/2014

Unless specified otherwise, all vector spaces can be real or complex, every L^p space is equipped with the corresponding L^p-norm ($1 \leq p \leq \infty$), and the space $\mathscr{C}[a,b]$ is equipped with the sup norm.

Problem 1. Suppose X, Y are normed spaces, $L_n \to L$ in $\mathscr{L}(X,Y)$ and $x_n \to x$ in X. Show that $L_n(x_n) \to L(x)$ in Y.

Problem 2. Suppose X is a normed space, Y is a Banach space, $\{L_n\}$ is a bounded sequence in $\mathscr{L}(X,Y)$, and $\{L_n(x)\}$ converges for every x in a dense subset of X.

(i) Show that $L(x) = \lim_{n\to\infty} L_n(x)$ exists for every $x \in X$, and $L \in \mathscr{L}(X,Y)$.

(ii) Show by an example that the convergence $L_n \to L$ in $\mathscr{L}(X,Y)$ may not hold.

(Hint for (ii): Look for a sequence $\{L_n\}$ of bounded linear functionals on $\mathscr{C}[0,1]$ with $\|L_n\| = 1$ such that $L_n(f) \to 0$ for every f.)

Problem 3. Consider the linear functionals Φ, Ψ on $\mathscr{C}[-1,1]$ defined by

$$\Phi(f) = \int_{-1}^{1} x f(x)\, dx$$

$$\Psi(f) = \int_{0}^{1} f(x)\, dx - \int_{-1}^{0} f(x)\, dx.$$

Show that Φ, Ψ are bounded and find their operator norms. How would the answers change if we regarded Φ, Ψ as linear functionals on $L^1[-1,1]$?

Problem 4. *The topology of pointwise convergence in $\mathscr{C}[0,1]$ is not normable:* Prove that there is no norm $\|\cdot\|$ on $\mathscr{C}[0,1]$ with respect to which $\|f_n - f\| \to 0$ iff $f_n(x) \to f(x)$ for every $x \in [0,1]$.

Problem 5. Suppose $(X, \|\cdot\|)$ is a normed space and Y is a proper closed subspace of X. Prove the following statements:

(i) $\|x + Y\| = \inf_{y\in Y} \|x + y\|$ defines a norm on the quotient space X/Y.

(ii) The natural projection $p : X \to X/Y$ is a bounded linear map with $\|p\| = 1$.

(iii) If X is a Banach space, so are Y and X/Y, and the natural projection p is an open map.

(iv) If Y and X/Y are Banach spaces, so is X.

Problem 6. It will be convenient to call a norm $\|\cdot\|$ on a vector space X a ***Banach norm*** if $(X, \|\cdot\|)$ is a Banach space.

(i) Show that if $\|\cdot\|$ and $\|\cdot\|'$ are Banach norms on X and $\|x\| \leq C\|x\|'$ for some constant $C > 0$, then $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms.

(ii) Critique the following "proof" that any two Banach norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space X are equivalent: Define $\|x\|'' = \|x\| + \|x\|'$, which is a Banach norm on X since every Cauchy sequence in $\|\cdot\|''$ is Cauchy in both $\|\cdot\|$ and $\|\cdot\|'$. Since $\|x\| \leq \|x\|''$ and $\|x\|' \leq \|x\|''$, part (i) shows that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent to $\|\cdot\|''$, hence equivalent to each other.

(Hint for (i): Use the open mapping theorem. With regards to (ii), one can show that every infinite dimensional Banach space admits non-equivalent Banach norms.)

Problem 7. Recall that ℓ^1 and ℓ^∞ are the vector spaces of all complex-valued sequences $x = \{x_i\}$ for which the norms

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i| \qquad \text{and} \qquad \|x\|_\infty = \sup_i |x_i|$$

are finite. Let c_0 be the subspace of ℓ^∞ consisting of all $x = \{x_i\}$ for which $\lim_{i\to\infty} x_i = 0$. Prove the following statements:

(i) Every $x \in \ell^\infty$ defines an $\hat{x} \in (\ell^1)^*$ by

$$\hat{x}(y) = \sum_{i=1}^{\infty} x_i y_i,$$

and $\|\hat{x}\| = \|x\|_\infty$. Moreover, every element of $(\ell^1)^*$ is of the form $\hat{x}$ for some $x \in \ell^\infty$. Thus, the map $x \mapsto \hat{x}$ is an isometric isomorphism between ℓ^∞ and $(\ell^1)^*$.

(ii) Every $x \in \ell^1$ defines an $\hat{x} \in (c_0)^*$ by

$$\hat{x}(y) = \sum_{i=1}^{\infty} x_i y_i,$$

and $\|\hat{x}\| = \|x\|_1$. Moreover, every element of $(c_0)^*$ is of the form $\hat{x}$ for some $x \in \ell^1$. Thus, the map $x \mapsto \hat{x}$ is an isometric isomorphism between ℓ^1 and $(c_0)^*$.

(iii) Every $x \in \ell^1$ defines an $\hat{x} \in (\ell^\infty)^*$ by the same formula as in (ii), and again $\|\hat{x}\| = \|x\|_1$. However, there are non-trivial elements of $(\ell^\infty)^*$ which vanish on c_0, so the map $x \mapsto \hat{x}$ from ℓ^1 to $(\ell^\infty)^*$ is not surjective.

(Hint: The space S of sequences $\{z_i\}$ where $z_i = 0$ for all but finitely many i is dense in both ℓ^1 and c_0, so every element of $(\ell^1)^*$ or $(c_0)^*$ is determined by its values on S.)

Problem 8.

(i) Show that ℓ^1 and c_0 are separable Banach spaces but ℓ^∞ is not.

(ii) Show, however, that every separable Banach space is isometric to a subspace of ℓ^∞. In other words, if $(X, \|\cdot\|)$ is a separable Banach space, there is an injective linear map $L : X \to \ell^\infty$ such that $\|L(x)\|_\infty = \|x\|$ for every $x \in X$.

(Hint for (ii): Take a countable dense set $\{x_n\}$ in X and find $f_n \in X^*$ such that $\|f_n\| = 1$ and $f_n(x_n) = \|x_n\|$. Now manufacture L using the f_n.)

Problem 9. A sequence $\{x_n\}$ in a normed space X ***converges weakly*** to $x \in X$, written as $x_n \overset{\text{w}}{\to} x$, if $f(x_n) \to f(x)$ for every $f \in X^*$. Prove the following statements:

(i) A sequence has at most one weak limit, and $x_n \to x$ implies $x_n \overset{\text{w}}{\to} x$.

(ii) If $x_n \overset{\text{w}}{\to} x$, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

(iii) If $x_n \overset{\text{w}}{\to} x$, then $\{x_n\}$ is a bounded sequence.

(Hint for (iii): Let $\hat{x}_n$ and $\hat{x}$ be the corresponding elements of X^{**}, so $\hat{x}_n(f) \to \hat{x}(f)$ for every $f \in X^*$. Apply the uniform boundedness principle to the collection $\{\hat{x}_n\}$.)

Problem 10. In this problem, you can use the isomorphisms $(c_0)^* \cong \ell^1$ and $(\ell^1)^* \cong \ell^\infty$ of problem 7.

(i) Find a sequence $\{x_n\}$ in c_0 such that $x_n \overset{\text{w}}{\to} 0$ but $\|x_n\|_\infty = 1$ for all n. This shows that a weakly convergent sequence is not necessarily convergent.

(ii) Prove the following special property of ℓ^1: If $x_n \overset{\text{w}}{\to} x$ in ℓ^1, then $x_n \to x$.