A Course in Complex Analysis

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CHAPTER 4

Möbius Maps and Hyperbolic Geometry

§4.1. The Möbius group Möb

We begin by studying a class of rational functions which play an important role in complex analysis.

**Definition 4.1.** A Möbius map (or fractional linear transformation) is a rational function of the form

\[ z \mapsto \frac{az + b}{cz + d}, \]

where \( a, b, c, d \in \mathbb{C} \) and \( ad - bc \neq 0 \).

Every Möbius map \( f(z) = \frac{az + b}{cz + d} \) is a one-to-one holomorphic map \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) with the inverse \( f^{-1}(z) = \frac{dz - b}{-cz + a} \) which is also Möbius. Note that \( f(\infty) = a/c \) or \( \infty \) according as \( c \neq 0 \) or \( c = 0 \) and \( f^{-1}(\infty) = -d/c \) or \( \infty \) according as \( c \neq 0 \) or \( c = 0 \). Thus, \( f \) defines a biholomorphism \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \).

It is easy to verify that the collection of all Möbius maps forms a group under composition, called the **Möbius group**, which we denote by \( \text{Möb} \).

Algebraically, Möb can be identified with a matrix group. There is an obvious way of associating to every non-singular \( 2 \times 2 \) matrix \( A \) over \( \mathbb{C} \) a Möbius map \( f_A \):

If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) then \( f_A(z) = \frac{az + b}{cz + d} \).

Under this association multiplication of matrices corresponds to composition of maps:

\[ f_{AB} = f_A \circ f_B. \]

This association, however, is far from one-to-one because simultaneous multiplication of the entries of a matrix by a non-zero complex number would yield the same map:

\[ f_{\lambda A} = f_A \quad \text{for all} \ \lambda \in \mathbb{C}^*. \]

We can almost rectify this non-uniqueness issue by replacing \( A \) with a multiple \( \lambda A \) so that \( \det(\lambda A) = \lambda^2(ad - bc) = 1 \). Since there are exactly two such multiples which only differ by a sign, we see that every Möbius map is of the form \( f_A \) for a matrix \( A \) with \( \det(A) = 1 \), and \( A \) is unique up to multiplication by \(-1\). Thus, Möb is
isomorphic to the quotient group
\[
\text{PSL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\} / \{ \pm I \}.
\]

There are special types of M"obius maps which will be utilized in the subsequent discussion:

- **translations** \( T_b : z \mapsto z + b \), with \( b \in \mathbb{C} \);
- **linear maps** \( L_a : z \mapsto az \), with \( a \in \mathbb{C}^* \); and
- the **inversion** across the unit circle: \( I : z \mapsto 1/z \).

It is easy to see that M"ob is generated by these special maps. Take any \( f : z \mapsto (az + b)/(cz + d) \) in M"ob and consider two cases: If \( c = 0 \), then
\[
f = T_{b'} \circ L_{a'}, \quad \text{where} \quad a' = \frac{a}{d} \quad \text{and} \quad b' = \frac{b}{d}.
\]

If \( c \neq 0 \), a brief computation shows that
\[
f(z) = \frac{a}{c} + \frac{bc - ad}{c^2} \frac{1}{z + \frac{d}{c}},
\]
so
\[
f = T_{d'} \circ L_{b'} \circ I \circ T_{d'},
\]
where
\[
a' = \frac{a}{c}, \quad b' = \frac{bc - ad}{c^2}, \quad \text{and} \quad d' = \frac{d}{c}.
\]

**Theorem 4.2** (Basic properties of M"ob).

(i) *The action of M"ob on \( \hat{\mathbb{C}} \) is "simply 3-transitive," that is, given triples \((p_1, p_2, p_3)\) and \((q_1, q_2, q_3)\) of distinct points in \( \hat{\mathbb{C}} \), there exists a unique element of M"ob which sends \( p_k \) to \( q_k \) for \( k = 1, 2, 3 \).*

(ii) *The action of M"ob preserves the family of circles in \( \hat{\mathbb{C}} \).*

In (ii), by a circle in \( \hat{\mathbb{C}} \) we mean the image of a Euclidean circle in \( \mathbb{S}^2 \subset \mathbb{R}^3 \) under the stereographic projection (compare §1.5). It is not hard to check that a circle in \( \hat{\mathbb{C}} \) is either a Euclidean circle or a straight line in the plane (which can be thought of as a circle passing through \( \infty \)); see problem [1]

**PROOF.** To prove (i), it suffices to show that every triple \((p_1, p_2, p_3)\) can be sent to \((0, 1, \infty)\) by a unique M"obius map. That there is such a map \( f \) is easy to see; simply take
\[
f(z) = \frac{(p_2 - p_3)}{(p_2 - p_1)} \frac{(z - p_1)}{(z - p_3)}.
\]
(If one of the $p_k$ is $\infty$, the formula for $f$ must be interpreted accordingly. Thus

$$f(z) = \frac{p_2 - p_3}{z - p_3} \quad \text{or} \quad \frac{z - p_1}{z - p_3} \quad \text{or} \quad \frac{z - p_1}{p_2 - p_1}$$

if $p_1$ or $p_2$ or $p_3$ is $\infty$, respectively.) To see uniqueness, suppose $g \in \text{Möb}$ sends $(p_1, p_2, p_3)$ to $(0, 1, \infty)$ also. Then $f \circ g^{-1}$ is a Möbius map which fixes 0, 1 and $\infty$. Writing $f \circ g^{-1}(z) = (az + b)/(cz + d)$ and imposing these three conditions, it follows that $b = c = 0$ and $a = d$, which gives $f \circ g^{-1}(z) = z$ or $f = g$.

To prove (ii), we need show that if $\Gamma$ is a Euclidean circle or line in $\mathbb{C}$, so is $f(\Gamma)$ for every $f \in \text{Möb}$. It suffices to verify this for the special maps $f = T_b, L_a$, and $\iota$ in (2) since they generate the whole group $\text{Möb}$. As $f(\Gamma)$ is clearly a circle or line when $f = T_b$ or $L_a$, we will only consider the case $f = \iota$. A line in $\mathbb{C}$ has an equation $ax + by + c = 0$, where $a, b, c$ are real and $a, b$ are not simultaneously zero. Changing to the complex-variable notation, this becomes

$$\bar{p}z + p\bar{z} + c = 0,$$

where $p = (a + ib)/2$ is a non-zero complex number and $c$ is real. A Euclidean circle of radius $r > 0$ centered at $p \in \mathbb{C}$ has an equation $|z - p| = r$ or $(z - p)(\bar{z} - \bar{p}) = r^2$, which can be written as

$$|z|^2 - (\bar{p}z + p\bar{z}) + |p|^2 - r^2 = 0.$$

Comparing (3) and (4), we see that every circle in $\hat{\mathbb{C}}$ has an equation of the form

$$t|z|^2 - (\bar{q}z + q\bar{z}) + s = 0,$$

where $t, s$ are real, $q$ is non-zero complex, and $ts < |q|^2$. The case $t = 0$ gives a line as in (3). The case $t \neq 0$ gives a circle centered at $p = q/t$ of radius $r = (\sqrt{|q|^2 - ts})/|t|$. Now, under the inversion $w = \iota(z) = 1/z$ the equation (5) transforms into

$$\frac{t}{|w|^2} - \left(\frac{\bar{q}}{w} + \frac{q}{\bar{w}}\right) + s = 0,$$

or

$$s|w|^2 - (qw + \bar{q}\bar{w}) + t = 0,$$

which is another equation of the form (5). This proves the assertion (ii). \qed
Example 4.3 (Communicated to me by W. Yassiyevich). Suppose $p_1, p_2, p_3, p_4$ are four distinct points lying on a circle $\Gamma$ in this counterclockwise order. The Möbius map $f(z) = 1/(z - p_1)$ sends $p_1$ to $\infty$, hence $\Gamma$ to a straight line $L = f(\Gamma)$. Let $q_k = f(p_k)$. Looking at the cyclic order of the points on $\Gamma$, we see that $q_3$ lies between $q_2$ and $q_4$ on $L$. Hence,

$$|q_2 - q_4| = |q_2 - q_3| + |q_3 - q_4|.$$  

Substituting $q_k = 1/(p_k - p_1)$ and simplifying, we obtain

$$|p_1 - p_3| |p_2 - p_4| = |p_1 - p_2| |p_3 - p_4| + |p_1 - p_4| |p_2 - p_3|$$

This is Ptolemy’s Theorem in Euclidean geometry: If a quadrilateral is inscribed in a circle, the sum of the products of its two pairs of opposite sides is equal to the product of its diagonals. Running this argument backwards, it follows that (6) is in fact equivalent to the $p_k$ lying on a circle.

The fact that Möbius maps send circles to circles and they preserve angles provides a simple but powerful tool to construct biholomorphisms between domains that are bounded by circles or circular arcs. Here is an example:

Example 4.4 (Cayley map). Let $\mathbb{H}$ be the upper half-plane $\{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$. We find a map $\phi \in \text{Möb}$ whose restriction to $\mathbb{H}$ gives a biholomorphism $\phi : \mathbb{H} \rightarrow \mathbb{D}$ (of course there are infinitely many such maps). We claim that the Möbius map $\phi$ for which

$$\phi(0) = 1 \quad \phi(i) = 0 \quad \phi(\infty) = -1$$

does the trick. In fact, $\phi$ sends the imaginary axis (the unique circle passing through $0, i, \infty$) to the real axis (the unique circle passing through $1, 0, -1$). The real axis, which is a circle passing through $0, \infty$ and meets the imaginary axis orthogonally at $0$, must map to a circle passing through $1, -1$ which meets the real axis orthogonally at $1$. The only such circle is the unit circle centered at the origin, hence $\phi(\mathbb{R} \cup \{\infty\}) = \mathbb{T}$. Since $\phi$ is a homeomorphism, the upper half-plane must map onto one of the components of $\mathbb{C} \setminus \mathbb{T}$. Since $\phi(i) = 0$, this component must be $\mathbb{D}$. This shows $\phi$ maps $\mathbb{H}$ biholomorphically onto $\mathbb{D}$.

Alternatively, one can easily check that the map $\phi \in \text{Möb}$ subject to the conditions (7) has the formula

$$\phi(z) = \frac{i - z}{i + z}$$

and that $\text{Im}(z) > 0$ if and only if $|\phi(z)| < 1$. The map $\phi$ is traditionally called the Cayley map (see Fig. 4).

§4.2. Three automorphism groups

**Definition 4.5.** Let $U$ be a non-empty open subset of $\hat{\mathbb{C}}$. The automorphism group $\text{Aut}(U)$ is the group of all biholomorphisms $U \rightarrow U$ under composition. Thus $f \in \text{Aut}(U)$ if and only if $f : U \rightarrow U$ is bijective and holomorphic.

Our main goal in this section is to determine $\text{Aut}(U)$ in the three simply connected cases $U = \hat{\mathbb{C}}, \mathbb{C}$ and $\mathbb{D}$. The importance of this issue will become apparent when we discuss uniformization of Riemann surfaces in chapter ??.
Figure 1. The Cayley map \( w = \phi(z) = (i - z)/(i + z) \) and its inverse \( z = \phi^{-1}(w) = i(1 - z)/(1 + z) \) define standard biholomorphisms between \( \mathbb{H} \) and \( \mathbb{D} \). Under \( \phi \) the horizontal lines in \( \mathbb{H} \) map to circles in \( \mathbb{D} \) that are tangent to \( \mathbb{T} \) at \(-1\), while the vertical lines in \( \mathbb{H} \) map to circular arcs in \( \mathbb{D} \) that are orthogonal to \( \mathbb{T} \) at \(-1\).

**Theorem 4.6.** The automorphism group \( \text{Aut}(\hat{\mathbb{C}}) \) of the sphere coincides with the Möbius group \( \text{Möb} \).

**Proof.** Since \( \text{Möb} \) is a subgroup of \( \text{Aut}(\hat{\mathbb{C}}) \), it suffices to check that every \( f \in \text{Aut}(\hat{\mathbb{C}}) \) is a Möbius map. Since the action of \( \text{Möb} \) is transitive, we can find \( g \in \text{Möb} \) such that \( (g \circ f)(\infty) = \infty \). By injectivity, it follows that \( (g \circ f)^{-1}(\infty) = \infty \). Since \( g \circ f \in \text{Aut}(\hat{\mathbb{C}}) \) is a rational map (Theorem 1.80), it must be a polynomial. But under a polynomial of degree \( d \), a generic point has \( d \) distinct preimages. Hence, as an injective polynomial, \( g \circ f \) must have degree 1, so \( g \circ f \) has the form \( h : z \mapsto az + b \in \text{Möb} \). It follows that \( f = g^{-1} \circ h \in \text{Möb} \). \( \square \)

**Theorem 4.7.** The automorphism group \( \text{Aut}(\mathbb{C}) \) of the plane is the subgroup of \( \text{Möb} \) consisting of all affine maps of the form \( z \mapsto az + b \) with \( a \in \mathbb{C}^* \) and \( b \in \mathbb{C} \).

Alternatively, \( \text{Aut}(\mathbb{C}) \) can be described as the subgroup of \( \text{Möb} \) consisting of all maps which fix the point \( \infty \).

**Proof.** Clearly every affine map is in \( \text{Aut}(\mathbb{C}) \). Let \( f \in \text{Aut}(\mathbb{C}) \). Extend \( f \) to a homeomorphism \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) by setting \( f(\infty) = \infty \). By Riemann’s Removable Singularity Theorem 1.68 applied to \( 1/(f(1/z)) \), the extended map is holomorphic in a neighborhood of \( \infty \), so \( f \in \text{Aut}(\hat{\mathbb{C}}) \). By Theorem 4.6 \( f \) is a Möbius map of the
form \( z \mapsto (az + b)/(cz + d) \). Since \( f(\infty) = \infty \), we have \( c = 0 \), which proves \( f \) is affine.

To determine the group \( \text{Aut}(\mathbb{D}) \), we first need the following important result, whose many variations and generalizations we will encounter later in the book.

**Theorem 4.8** (Schwarz Lemma, 1869). Suppose \( f : \mathbb{D} \to \mathbb{D} \) is holomorphic and \( f(0) = 0 \). Then

\[
|f(z)| \leq |z|, \quad z \in \mathbb{D}
\]

and

\[
|f'(0)| \leq 1.
\]

If \( |f(z)| = |z| \) for some \( z \in \mathbb{D}^* \), or if \( |f'(0)| = 1 \), then \( f \) is a rigid rotation of the form \( f(z) = \lambda z \), where \( |\lambda| = 1 \).

Note that the inequality \( |f'(0)| \leq 1 \), even without the assumption \( f(0) = 0 \), follows from Theorem [1.42].

**PROOF.** Let \( f(z) = \sum_{k=1}^{\infty} a_k z^k \) be the power series representation of \( f \), where the constant term \( a_0 \) is missing since \( f(0) = 0 \). Define \( g(z) = f(z)/z = \sum_{k=0}^{\infty} a_{k+1} z^k \), which has a removable singularity at the origin and hence \( g \in \mathcal{O}(\mathbb{D}) \). If \( 0 < |z| = r < 1 \), then

\[
|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}.
\]

By the Maximum Principle,

\[
|g(z)| \leq \frac{1}{r} \quad \text{whenever } |z| \leq r.
\]

Letting \( r \to 1 \), it follows that

\[
|g(z)| \leq 1, \quad z \in \mathbb{D}.
\]

This proves (8) and (9) simultaneously, since \( f'(0) = g(0) \).

If \( |f(z)| = |z| \) for some \( z \in \mathbb{D}^* \), or if \( |f'(0)| = 1 \), then \( |g(z)| = 1 \) for some \( z \in \mathbb{D} \). Hence, by the Maximum Principle, \( g \) is a constant \( \lambda \) with \( |\lambda| = 1 \), which shows \( f(z) = \lambda z \) for all \( z \in \mathbb{D} \).

**Definition 4.9.** If \( a \in \mathbb{C} \) and \( |a| \neq 1 \), we denote by \( \phi_a \) the Möbius map

\[
\phi_a(z) = \frac{z - a}{1 - \overline{a}z}.
\]

The following properties are easily verified:

(i) \( \phi_a(a) = 0 \) and \( \phi_a(0) = -a \);
(ii) \( \phi'_a(z) = (1 - |a|^2)/(1 - \bar{a}z)^2 \), hence
\[
\phi'_a(0) = 1 - |a|^2 \quad \text{and} \quad \phi'_a(a) = \frac{1}{1 - |a|^2};
\]

(iii) \((\phi_a)^{-1} = \phi_{-a} : z \mapsto (z + a)/(1 + \bar{a}z)\);

(iv) If \( a \in \mathbb{D} \), then \( \phi_a \in \text{Aut}(\mathbb{D}) \).

Only (iv) requires a comment: If \( |z| = 1 \), then
\[
|\phi_a(z)| = \frac{|z - a|}{|1 - \bar{a}z|} = \frac{|z - a|}{|z| |\bar{z} - \bar{a}|} = 1.
\]

Thus \( \phi_a \) maps the unit circle onto the unit circle. Since \( \phi_a \) is a homeomorphism of the sphere, it should map \( \mathbb{D} \) bijectively either to \( \mathbb{D} \) or to \( \mathbb{C} \setminus \mathbb{D} \). The assumption \( a \in \mathbb{D} \) shows that \( \phi_a(0) = -a \in \mathbb{D} \). Hence \( \phi_a(\mathbb{D}) = \mathbb{D} \).

**Theorem 4.10.** The automorphism group \( \text{Aut}(\mathbb{D}) \) of the unit disk is the subgroup of \( \text{M"{o}b} \) consisting of all maps of the form
\[
z \mapsto \lambda \frac{z - a}{1 - \bar{a}z},
\]
where \( a \in \mathbb{D} \) and \( |\lambda| = 1 \).

**Proof.** Every Möbius map of the above form is a rotation of \( \phi_a \) for some \( a \in \mathbb{D} \), hence an automorphism of the disk. Conversely, suppose \( f \in \text{Aut}(\mathbb{D}) \) and let \( a = f^{-1}(0) \in \mathbb{D} \). Then \( g = f \circ \phi_{-a} \) is in \( \text{Aut}(\mathbb{D}) \) and \( g(0) = 0 \). By the Schwarz Lemma, \( |g'(0)| \leq 1 \). But \( g^{-1} \in \text{Aut}(\mathbb{D}) \) as well, and the Schwarz Lemma applied to \( g^{-1} \) gives \( |(g^{-1})'(0)| = 1/|g'(0)| \leq 1 \). Thus \( |g'(0)| = 1 \) and so \( g(z) = \lambda z \) for some constant \( \lambda \) with \( |\lambda| = 1 \). It follows that \( f = \lambda \phi_a \), as required.

Here is a variation of Theorem 4.8 in which the assumption \( f(0) = 0 \) is removed:

**Theorem 4.11.** Suppose \( f : \mathbb{D} \to \mathbb{D} \) is holomorphic. Then
\[
|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \quad z \in \mathbb{D}.
\]

Equality holds at some \( z \in \mathbb{D} \) if and only if \( f \in \text{Aut}(\mathbb{D}) \).

**Proof.** Take any \( a \in \mathbb{D} \) and let \( b = f(a) \). The holomorphic function
\[
g = \phi_b \circ f \circ \phi_{-a}
\]
fixes the origin, hence by the Schwarz Lemma \( |g'(0)| \leq 1 \), or
\[
|\phi'_b(b)| |f'(a)| |\phi'_{-a}(0)| \leq 1.
\]
Since \( \phi'_b(b) = 1/(1 - |b|^2) \) and \( |\phi'_{-a}(0)| = 1 - |a|^2 \), this proves
\[
|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}.
\]
Equality occurs at \( a \in \mathbb{D} \) if and only if \( |g'(0)| = 1 \) for the function \( g \) above, which is the case if and only if \( g \) is a rotation. This happens precisely when \( f = \phi_{-b} \circ g \circ \phi_a \in \text{Aut}(\mathbb{D}) \).

Problems

(1) Let \( \phi : S^2 \to \hat{\mathbb{C}} \) be the stereographic projection defined by \((23)\) in §1.5. Verify the following statements:

(i) \( \phi \) maps every Euclidean circle on \( S^2 \) (i.e., the intersection of a plane with \( S^2 \)) to a Euclidean circle or line in \( \mathbb{C} \).

(ii) For \( z, w \in \hat{\mathbb{C}} \), the points \( \phi^{-1}(z), \phi^{-1}(w) \) are diametrically opposite (i.e., \( \phi^{-1}(z) = -\phi^{-1}(w) \)) if and only if \( z = -1/\bar{w} \).

(iii) The chordal distance on \( S^2 \) given by
\[
\text{dist}((x_1, x_2, x_3), (y_1, y_2, y_3)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}
\]
corresponds under \( \phi \) to the distance
\[
\text{dist}(z, w) = \begin{cases} 
\frac{2|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}} & z, w \in \mathbb{C} \\
\frac{2}{\sqrt{1 + |z|^2}} & z \in \mathbb{C}, w = \infty 
\end{cases}
\]
on the Riemann sphere. In the infinitesimal form, the latter distance is described by the Riemannian metric
\[
\frac{2}{(1 + |z|^2)}|dz|
\]
often known as the spherical metric.

(2) Which elements of \( \text{M"ob} \) correspond to rigid rotations of the sphere \( S^2 \) under the stereographic projection?

(3) Define the cross ratio of an ordered quadruple \((a, b, c, d)\) of distinct points in \( \hat{\mathbb{C}} \) by
\[
\chi(a, b, c, d) = \frac{(a - b)(c - d)}{(a - d)(c - b)}.
\]
If one of the points is \( \infty \), the definition extends by continuity; thus
\[
\chi(\infty, b, c, d) = \frac{c - d}{c - b}, \quad \chi(a, \infty, c, d) = \frac{c - d}{a - d},
\]
and so on. Verify the following statements:

(i) If \( f \in \text{Aut}(\mathbb{C}) \) is the unique transformation which maps \((b, c, d)\) to \((0, 1, \infty)\), then \( \chi(a, b, c, d) = f(a) \).

(ii) If \( f \in \text{Aut}(\hat{\mathbb{C}}) \), then
\[
\chi(a, b, c, d) = \chi(f(a), f(b), f(c), f(d))
\]
for every quadruple \((a, b, c, d)\). Conversely, if a homeomorphism \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) preserves cross ratios of all quadruples, it must be an element of \( \text{Aut}(\hat{\mathbb{C}}) \).

(iii) Four points \( a, b, c, d \) lie on a circle in \( \hat{\mathbb{C}} \) if and only if \( \chi(a, b, c, d) \) is real.
Determine the automorphism group of the doubly-punctured plane \( \widehat{\mathbb{C}} \).

We say that \( f, g \in \text{Aut}(\widehat{\mathbb{C}}) \) are conjugate if there exists \( \varphi \in \text{Aut}(\widehat{\mathbb{C}}) \) such that \( \varphi \circ f = g \circ \varphi \).

(i) Show that every non-identity \( f \in \text{Aut}(\widehat{\mathbb{C}}) \) has one or two fixed points in \( \hat{\mathbb{C}} \), and \( f \) and \( g \) have the same number of fixed points if they are conjugate.

(ii) Show that every \( f \in \text{Aut}(\widehat{\mathbb{C}}) \) with one fixed point is conjugate to the translation \( z \mapsto z + 1 \). On the other hand, if \( f \) has two fixed points, then it is conjugate to a dilation \( z \mapsto \lambda z \) for some \( \lambda \in \mathbb{C} \setminus \{0, 1\} \). In this case, show that the number \( \tau(f) = \lambda + \lambda^{-1} \) is uniquely determined by \( f \).

(iii) Conclude that if \( f \) and \( g \) each have one fixed point, they are conjugate, while if they each have two fixed points, they are conjugate if and only if \( \tau(f) = \tau(g) \).

Let \( \mathbb{H} = \{ z = x + iy \in \mathbb{C} : y > 0 \} \) denote the upper half-plane. Using the fact that \( \phi(z) = (i - z)/(i + z) \) maps \( \mathbb{H} \) biholomorphically to \( \mathbb{D} \) and your knowledge of \( \text{Aut}(\mathbb{D}) \), show that

\[
\text{Aut}(\mathbb{H}) = \{ z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R}, ad - bc > 0 \}.
\]

Under the isomorphism \( \text{Aut}(\mathbb{H}) \cong \text{Aut}(\mathbb{D}) \) induced by \( \phi \), find the element in \( \text{Aut}(\mathbb{H}) \) that corresponds to the rotation \( z \mapsto e^{i\theta}z \) in \( \text{Aut}(\mathbb{D}) \).

Determine the automorphism group of the doubly-punctured plane \( \mathbb{C} \setminus \{0, 1\} \). What can you say about the automorphism group of the \( n \)-punctured plane \( \mathbb{C} \setminus \{p_1, \ldots, p_n\} \), where the \( p_k \) are distinct? What condition(s) on the points \( p_k \) will guarantee that this automorphism group is non-trivial?

Suppose \( f \in \text{Aut}(\widehat{\mathbb{C}}) \) does not fix \( \infty \), so \( f(z) = (az + b)/(cz + d) \) with \( ad - bc = 1 \) and \( c \neq 0 \).

(i) Show that the locus \( \Gamma_f = \{ z \in \mathbb{C} : \left| f'(z) \right| = 1 \} \) is a Euclidean circle and find its center and radius. This is called the isometric circle of \( f \) since \( f \) preserves the Euclidean length of tangent vectors at the points of \( \Gamma_f \).

(ii) Verify that \( \left| f'(z) \right| > 1 \) for \( z \) in the interior of \( \Gamma_f \) and \( \left| f'(z) \right| < 1 \) for \( z \) in the exterior of \( \Gamma_f \).

(iii) Conclude, without any computation, that \( f(\Gamma_f) = \Gamma_{f^{-1}} \).

A rational function of the form

\[
B(z) = \lambda z^m \left( \frac{z - a_1}{1 - \overline{a_1}z} \right) \cdots \left( \frac{z - a_n}{1 - \overline{a_n}z} \right),
\]

in which \( |\lambda| = 1 \), \( m \in \mathbb{Z} \), and \( |a_k| \neq 1 \) for all \( 1 \leq k \leq n \), is called a finite Blaschke product.

(i) Show that any such \( B \) satisfies \( B(1/\overline{z}) = 1/\overline{B(z)} \) for all \( z \). In other words, \( B \) commutes with the reflection \( z \mapsto 1/\overline{z} \). In particular, \( z \) is a zero of \( B \) if and only if \( 1/\overline{z} \) is a pole of \( B \), and \( |B(z)| = 1 \) whenever \( |z| = 1 \).

(ii) Show that if all the zeros of a finite Blaschke product \( B \) belong to \( \mathbb{D} \), then \( B(\mathbb{D}) \subset \mathbb{D} \) and \( B(\hat{\mathbb{C}} \setminus \mathbb{D}) \subset \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \).

(iii) Show that \( z \) is a critical point of \( B \) if and only if \( 1/\overline{z} \) is a critical point of \( B \).
(9) Suppose $f$ is a rational function such that $|f(z)| = 1$ whenever $|z| = 1$. Show that $f$ is a finite Blaschke product. (Hint: First show that $\frac{f(1/z)}{z}$ is a rational function which agrees with $1/f(z)$ on the unit circle, hence everywhere.)

(10) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and $f(a_k) = 0$ for $k = 1, \ldots, n$. Show that for all $z \in \mathbb{D}$,

$$|f(z)| \leq \prod_{k=1}^{n} \left| \frac{z - a_k}{1 - \overline{a_k}z} \right|.$$ 

What can you say about $f$ if equality holds for some $z \notin \{a_1, \ldots, a_n\}$?

(11) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, with $f(0) = f(1/2) = f(-1/2) = 0$. Show that

$$\left| f \left( \frac{1}{4} \right) \right| \leq \frac{1}{21}.$$ 

Show that the bound $1/21$ cannot be made smaller.

(12) Let $P$ be a complex polynomial of degree $n \geq 1$ whose roots belong to the open unit disk. Define the polynomial $Q$ by $Q(z) = z^n \overline{P(1/z)}$. Show that the roots of the polynomial $P(z) + Q(z)$ belong to the unit circle.

(13) Let $f \in \mathcal{O}(\mathbb{D})$ satisfy $f(0) = 0$ and $|f(z) + zf'(z)| < 1$ for all $z \in \mathbb{D}$. Show that $|f(z)| < 1$ for all $z \in \mathbb{D}$.

(14) If $f \in \mathcal{O}(\mathbb{D})$, show that there exists a sequence $\{z_k\}$ in $\mathbb{D}$, with $|z_k| \to 1$ as $k \to \infty$, such that $\{f(z_k)\}$ is a bounded sequence in $\mathbb{C}$.
§5.1. The Riemann Mapping Theorem

Theorem 5.1 (Riemann Mapping Theorem, 1851). Every simply connected domain in $\mathbb{C}$, except $\mathbb{C}$ itself, is biholomorphic to $D$.

Clearly $\mathbb{C}$ is not biholomorphic to $D$ because every holomorphic map $\mathbb{C} \to D$ is constant by Liouville’s Theorem. The theorem can be expressed more invariantly by saying that every simply connected domain in $\mathbb{C}$ which misses at least two (hence infinitely many) points is biholomorphic to $D$.

Corollary 5.2. If $U \subset \mathbb{C}$ is a simply connected domain and $p \in U$, then there exists a unique biholomorphism $f : U \to D$, with $f(p) = 0$ and $f'(p) > 0$.

The inverse map $f^{-1} : D \to U$ is often called the normalized Riemann map of the pair $(U, p)$.

Proof. Take any biholomorphism $g : U \to D$ given by the Riemann Mapping Theorem. If $a = g(p) \in D$, the composition $h = \phi_a \circ g : U \to D$ is a biholomorphism with $h(p) = 0$. Here, as in Definition 4.9, $\phi_a$ denotes the disk automorphism $z \mapsto (z - a)/(1 - \bar{a}z)$. Let $\lambda = |h'(p)|/h'(p)$ so $|\lambda| = 1$ (note that $h'(p) \neq 0$ since $h$ is injective). The map $f = \lambda h : U \to D$ with be the required biholomorphism.

To see uniqueness, suppose $f$ and $\hat{f}$ are biholomorphisms $U \to D$, both sending $p$ to 0, and both having positive derivative at $p$. Then $\hat{f} \circ f^{-1}$ will be an automorphism of the disk which fixes 0 and has positive derivative there. It follows from the Schwarz Lemma that $\hat{f} \circ f^{-1}$ is the identity map of $D$, so $f = \hat{f}$. □

Proof of Theorem 5.1. Let $U \subset \mathbb{C}$ be a simply connected domain and fix some $p \in U$. Define

$$\mathcal{F} = \{ f : U \to D : f \text{ is holomorphic and injective and } f(p) = 0 \}.$$  

We will find a biholomorphism $U \to D$ in $\mathcal{F}$ as the solution to an extremal problem. This will be done in several steps:

Step 1. $\mathcal{F}$ is nonempty. Since $U \neq \mathbb{C}$, we can choose some $q \in \mathbb{C} \setminus U$. Since $U$ is simply connected, Corollary 2.17 shows that the non-vanishing function $z \mapsto z - q$ has a holomorphic square root $\hat{h}$ in $U$. Thus, $h \in \mathcal{O}(U)$ and $(h(z))^2 = z - q$ for all
$z \in U$. Evidently $h$ is injective. In particular, $h(U)$ is a domain in $\mathbb{C}$. Moreover, the domains $h(U)$ and $-h(U)$ are disjoint since if $h(z) = -h(w)$ for some $z, w \in U$, then $z - w = h^2(z) = h^2(w) = w - q$, or $z = w$. This shows $h(z) = 0$, or $z = q$, which is a contradiction since $q \notin U$. Now take a disk $\mathbb{D}(a, r) \subset -h(U)$, so $h(U) \cap \mathbb{D}(a, r) = \emptyset$. The Möbius map $\phi(z) = r/(z - a)$ maps $\mathbb{D}(a, r)$ to the complement of the closed unit disk $\mathbb{D}$, hence $h(U)$ into the unit disk $\mathbb{D}$. If $\psi$ is the automorphism of $\mathbb{D}$ which sends $r/(h(p) - a)$ to 0, it follows that $\psi \circ \phi \circ h \in \mathcal{F}$.

**Step 2.** If $f \in \mathcal{F}$ and $f(U) \subsetneq \mathbb{D}$, then there exists a $g \in \mathcal{F}$ such that $|g'(p)| > |f'(p)|$. This can be deduced most easily from the square root trick of Carathéodory and Koebe as follows. Take a point $a \in \mathbb{D} \setminus f(U)$. Then $\phi_a \circ f : U \to \mathbb{D}$ is injective and non-vanishing. (Here, as in Definition 4.9, $\phi_a$ denotes the automorphism of $\mathbb{D}$ defined by $\phi_a(z) = (z - a)/(1 - \bar{a}z)$ whose inverse $(\phi_a)^{-1}$ is $\phi_{-a}$. Since $U$ is simply connected, Corollary 2.17 shows there is an $h \in \mathcal{O}(U)$ such that $h^2 = \phi_a \circ f$. It is easy to see that $h$ is injective and $h(U) \subset \mathbb{D}$. If $b = h(p)$, it follows that $\mathcal{F}$

$$f = \phi_{-a} \circ s \circ h = (\phi_{-a} \circ s \circ \phi_{-b}) \circ g.$$

Note that $\phi_{-a} \circ s \circ \phi_{-b} : \mathbb{D} \to \mathbb{D}$ fixes the origin, and it is not injective because $\phi_{-a}$ and $\phi_{-b}$ are injective while $s$ is not. Hence, by the Schwarz Lemma, the strict inequality $|(\phi_{-a} \circ s \circ \phi_{-b})'(0)| < 1$ must hold. It follows that

$$|f'(p)| = |(\phi_{-a} \circ s \circ \phi_{-b})'(0)| \cdot |g'(p)| < |g'(p)|.$$

**Step 3.** Define

$$\sigma = \sup_{f \in \mathcal{F}} |f'(p)|.$$

Take a sequence $\{f_k\}$ in $\mathcal{F}$ such that $|f_k'(p)| \to \sigma$. Since the family $\mathcal{F}$ is uniformly bounded (by the constant 1), Montel’s Theorem 3.16 shows that there exists a subsequence $\{f_{k_j}\}$ which converges compactly in $U$ to some $f \in \mathcal{O}(U)$ as $j \to \infty$. Evidently $f(p) = 0$ and $|f'(p)| = \sigma$; in particular $0 < \sigma < +\infty$.

**Step 4.** We clearly have $f(U) \subset \mathbb{D}$. If $f$ takes a value on $\partial \mathbb{D}$, then by the Open Mapping Theorem, it must be constant. But $|f'(p)| = \sigma > 0$ shows this is not the case. Hence $f(U) \subset \mathbb{D}$. Similarly, since each $f_{k_j}$ is injective in $U$ and $f$ is non-constant, Theorem 3.15 shows that $f$ is injective. Thus $f \in \mathcal{F}$. By Step 2, $f(U) = \mathbb{D}$, so $f$ is a biholomorphism $U \to \mathbb{D}$.

Here are two topological corollaries of the Riemann Mapping Theorem.

**Corollary 5.3.** Every simply connected domain in the plane is homeomorphic to the open unit disk $\mathbb{D}$.

**Corollary 5.4.** If $U \subset \hat{\mathbb{C}}$ is a simply connected domain, then $\hat{\mathbb{C}} \setminus U$ is connected.
The converse statement is also true, that is, a domain in \( \hat{\mathbb{C}} \) with connected complement must be simply connected; see ??.

Note that this corollary is false if \( \hat{\mathbb{C}} \) is replaced with \( \mathbb{C} \). For example, the doubly-slit plane

\[
U = \mathbb{C} \setminus \{x \in \mathbb{R} : |x| \geq 1\}
\]
is simply connected, but \( \mathbb{C} \setminus U \) has two connected components.

**Proof.** If \( U = \hat{\mathbb{C}} \) or \( \hat{\mathbb{C}} \setminus \{\text{point}\} \), the result is clear. Otherwise, the Riemann Mapping Theorem shows there is a biholomorphism \( f : \mathbb{D} \to U \). For each \( 0 < r < 1 \), the set \( f(\mathbb{D}(0, r)) \) is a Jordan domain, hence its complement is compact and connected.

It follows that the nested intersection

\[
\hat{\mathbb{C}} \setminus U = \bigcap_{0 < r < 1} \left( \hat{\mathbb{C}} \setminus f(\mathbb{D}(0, r)) \right)
\]
is also compact and connected. □

A natural question is when the biholomorphism between a simply connected domain and the unit disk extends continuously to the boundary. The complete answer, provided by Carathéodory, depends directly on the topology of the boundary \( \partial U \).

Recall that a set \( X \subset \hat{\mathbb{C}} \) is **locally connected** if every point of \( X \) has arbitrarily small connected neighborhoods. More precisely, if for every \( p \in X \) and every open neighborhood \( U \) of \( p \) there is an open neighborhood \( V \subset U \) of \( p \) such that \( V \cap X \) is connected (compare Fig. ??). When \( X \) is compact, local connectivity is equivalent to the following condition: For every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that every pair \( p, q \in X \) whose distance is \( < \delta \) are contained in a connected set \( E \subset X \) whose diameter is \( < \varepsilon \) ?.

**Theorem 5.5** (Carathéodory, 1913). Suppose \( U \subset \hat{\mathbb{C}} \) is a simply connected domain which misses at least two points. Let \( f : \mathbb{D} \to U \) be any biholomorphism. Then

(i) \( f \) extends continuously to a map \( \overline{\mathbb{D}} \to \overline{U} \) if and only if \( \partial U \) is locally connected.

(ii) \( f \) extends homeomorphically to a map \( \overline{\mathbb{D}} \to \overline{U} \) if and only if \( \partial U \) is a Jordan curve.

The following classical result is an immediate consequence:

**Corollary 5.6** (Schönflies, 1902??). Let \( \Gamma \subset \hat{\mathbb{C}} \) be a Jordan curve and \( \varphi : \mathbb{T} \to \Gamma \) be any homeomorphism. Then \( \varphi \) extends to a homeomorphism \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \).

**Proof.** By the Jordan Curve Theorem, \( \hat{\mathbb{C}} \setminus \Gamma \) has two components \( U \) and \( V \), both simply connected, with \( \partial U = \partial V = \Gamma \). Take biholomorphisms \( f : \mathbb{D} \to U \) and \( g : \mathbb{D} \to V \). By the Carathéodory Theorem 5.5 both \( f \) and \( g \) extend homeomorphically to the closures. The maps \( f^{-1} \circ \varphi : \mathbb{T} \to \mathbb{T} \) and \( g^{-1} \circ \varphi \circ \iota : \mathbb{T} \to \mathbb{T} \) are thus circle homeomorphisms. Here \( \iota(z) = 1/z \). As such, they extend to homeomorphisms
Conformal Mappings

$F : \overline{D} \to \overline{D}$ and $G : \overline{D} \to \overline{D}$ (take, for example, the radial extensions). The map

$$\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

defined by

$$\Phi = \begin{cases} f \circ F & \text{inside} \ D \\ g \circ G \circ \iota & \text{outside} \ D \end{cases}$$

is now a homeomorphic extension of $\varphi$. \hfill \Box

§5.2. Schlicht Functions

**Definition 5.7** (Schlicht functions). A holomorphic function $f : \mathbb{D} \to \mathbb{C}$ is called *schlicht* if it is injective and normalized in the sense that $f(0) = 0$, $f'(0) = 1$. The class of all schlicht functions is denoted by $\mathcal{S}$.

Thus, every $f \in \mathcal{S}$ has a power series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad z \in \mathbb{D}.$$  

Note that by injectivity, $f : \mathbb{D} \to f(\mathbb{D})$ is a biholomorphism (Corollary 1.58).

**Example 5.8** (The Koebe function). The function $f : \mathbb{D} \to \mathbb{C}$ defined by

$$f(z) = z + \sum_{k=2}^{\infty} k z^k = \frac{z}{(1-z)^2}$$

is schlicht (compare Example 1.21). It is easy to verify directly that $f$ is injective in $\mathbb{D}$ but it is less trivial to show that the image domain $f(\mathbb{D})$ is the slit-plane $U = \mathbb{C} \setminus (-\infty, -1/4]$. Perhaps the most natural way of seeing this is by showing that $f$ is the normalized Riemann map of $(U, 0)$. To see this, consider the composition of the following conformal maps: First map $\mathbb{D}$ conformally to the right half-plane by the Möbius map

$$\zeta = \frac{1}{2} \left( \frac{1+z}{1-z} \right)$$

which sends 0 to 1/2. Then send the right half-plane conformally to the slit-plane $V = \mathbb{C} \setminus (-\infty, 0]$ by the squaring map $\xi = \zeta^2$, which sends 1/2 to 1/4. Finally, shift the slit plane by the map $w = \xi - 1/4$ which sends 1/4 to 0. The composition $z \mapsto \zeta \mapsto \xi \mapsto w$ is a biholomorphism $\mathbb{D} \to U$. It has the formula

$$w = \xi - \frac{1}{4} = \xi^2 - \frac{1}{4} = \frac{1}{4} \left( \frac{1+z}{1-z} \right)^2 - 1 = \frac{z}{(1-z)^2},$$

which is the Koebe function.

The study of schlicht functions is facilitated by the following construction: If $f \in \mathcal{S}$, the image $f(\mathbb{D})$ is a simply connected domain which may be unbounded and it may have unbounded complement. However, if we change coordinates using the map $w = \iota(z) = 1/z$, we obtain a biholomorphism $\psi_f = \iota \circ f \circ \iota : \hat{\mathbb{C}} \to \overline{\mathbb{D}} \to U$, where $U = \iota(f(\mathbb{D}))$ is a simply connected domain in $\hat{\mathbb{C}}$ containing $\infty$. The advantage
of this construction is that in the new coordinate $w$, the complement $\hat{\mathbb{C}} \setminus U$ is a compact subset of the plane.

For our future reference, let us find the relation between the power series of $f \in S$ and the Laurent series of its conjugate map $\psi_f$. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$
\psi_f(w) = \frac{1}{f(1/w)} = \frac{1}{\frac{1}{w} + \frac{a_2}{w^2} + \frac{a_3}{w^3} + \cdots}
= \frac{w}{1 + \frac{a_2}{w} + \frac{a_3}{w^2} + \cdots}
= w \left(1 - \frac{a_2}{w} + \frac{a_2^2 - a_3}{w^2} + \cdots\right),
$$

which gives

$$
(1) \quad \psi_f(w) = w - a_2 + \frac{a_2^2 - a_3}{w} + \cdots
$$

Example 5.9. Joukowski’s map $\psi(z) = z + 1/z$ and its properties.

**Theorem 5.10** (Gronwall’s Area Theorem, 1914). Suppose $U \subset \hat{\mathbb{C}}$ is a simply connected domain containing $\infty$, with $\hat{\mathbb{C}} \setminus U$ containing more than one point. Let $\psi : \hat{\mathbb{C}} \setminus \hat{\mathbb{D}} \to U$ be a biholomorphism fixing $\infty$, with the Laurent series

$$
\psi(w) = w + \sum_{k=0}^{\infty} b_k w^{-k} = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots
$$

Then,

$$
\text{area}(\hat{\mathbb{C}} \setminus U) = \pi \left(1 - \sum_{k=0}^{\infty} k |b_k|^2\right).
$$

In particular,

$$
(2) \quad |b_1| \leq 1.
$$

Equality occurs if and only if $\hat{\mathbb{C}} \setminus U$ is a straight segment of length 4 centered at $b_0$.

**Proof.** For $r > 1$, let $K_r$ be the compact set $\hat{\mathbb{C}} \setminus \psi(\{w : |w| > r\})$ bounded by the Jordan curve $\psi(\mathbb{T}(0, r))$. By Green’s Theorem,

$$
\text{area}(K_r) = \frac{1}{2i} \int_{\psi(\mathbb{T}(0, r))} \bar{z} \, dz
$$

For simplicity, write $\psi(w) = \sum_{k=-1}^{\infty} b_k w^{-k}$, where $b_{-1} = 1$. Taking the parametrization $z = \psi(re^{it}) = \sum_{k=-1}^{\infty} b_k r^{-k} e^{-ikt}$, $t \in [0, 2\pi]$, for the Jordan
The integral \( \int_0^{2\pi} e^{i(k-m)t} \, dt \) is zero if \( k \neq m \) and is \( 2\pi \) if \( k = m \). Hence,

\[
\text{area}(K_r) = -\pi \sum_{k=-\infty}^{\infty} k |b_k|^2 r^{-2k}.
\]

But \( \hat{C} \setminus U = \bigcap_{r>1} K_r \) and \( K_r \subset K_s \) whenever \( 1 < r < s \). Hence

\[
\text{area}(\hat{C} \setminus U) = \lim_{r \to 1} \text{area}(K_r) = -\pi \sum_{k=1}^{\infty} k |b_k|^2 = \pi \left( 1 - \sum_{k=0}^{\infty} k |b_k|^2 \right).
\]

The inequality (2) follows immediately from the above formula and the fact that \( \text{area}(\hat{C} \setminus U) \geq 0 \).

The equality \(|b_1| = 1\) occurs if and only if \( b_k = 0 \) for all \( k \geq 2 \), in which case \( \psi \) takes the form

\[
\psi(w) = w + b_0 + \frac{b_1}{w}.
\]

Write \( b_1 = e^{2i\theta} \) and consider the affine maps

\[
A(z) = e^{i\theta} z \quad \text{and} \quad B(z) = e^{-i\theta} (z - b_0).
\]

Then

\[
(B \circ \psi \circ A)(z) = z + \frac{1}{z}
\]

which maps \( \hat{C} \setminus \overline{D} \) conformally onto \( \hat{C} \setminus [-2, 2] \) (Example 5.9). Since \( A \) is a rotation, we conclude that \( \hat{C} \setminus U \) is the straight segment

\[
B^{-1}([-2, 2]) = [-2e^{i\theta} + b_0, 2e^{i\theta} + b_0].
\]

**Corollary 5.11.** If \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \) is in \( S \), then \( |a_2^2 - a_3| \leq 1 \).

**Proof.** Apply the inequality (2) to the conjugate biholomorphism \( \psi_f = t \circ f \circ t \) which by (1) has the Laurent series \( \psi_f(w) = w - a_2 + (a_2^2 - a_3)/w + \cdots \).

**Theorem 5.12** (Bieberbach inequality, 19??). If \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \) is in \( S \), then \( |a_2| \leq 2 \). Equality occurs if and only if \( f \) is conjugate to the Koebe function by a rotation, so \( f(D) = \mathbb{C} \setminus [e^{i\theta}/4, \infty) \) for some \( \theta \).
PROOF. Write \( f(z) = zg(z) \), where \( g(z) = 1 + \sum_{k=1}^{\infty} a_k + 1 z^k \) is holomorphic in \( \mathbb{D} \). Since \( f \neq 0 \) in \( \mathbb{D}^* \), we have \( g \neq 0 \) in \( \mathbb{D} \). Hence, \( g \) has a holomorphic square root \( h \in \mathcal{O}(\mathbb{D}) \) with \( h(0) = 1 \). Define

\[
g(z) = z h(z^2).
\]

Since \( f(z^2) = z^2 g(z^2) = z^2 (h(z^2))^2 \), we have

\[
f(z^2) = (g(z))^2.
\]

We check that \( g \in S \). Clearly \( g \in \mathcal{O}(\mathbb{D}) \), \( g(0) = 0 \), \( g'(0) = h(0) = 1 \). To show injectivity of \( g \), suppose \( g(z) = g(w) \) for some \( z, w \in \mathbb{D} \). Then by (4), \( f(z^2) = f(w^2) \), hence \( z = \pm w \) since \( f \) is injective. If \( z = w \), we are done. If \( z = -w \), then by (3), \( g(z) = -g(w) \). Hence \( g(z) = g(w) = 0 \), which by another application of (4) gives \( z = w = 0 \).

Now by (3), \( g \) has a power series representation \( g(z) = z + c z^3 + \cdots \) containing odd powers of \( z \) only. Comparing this with that of \( f \) using (4), we see that \( c = a_2 / 2 \), hence

\[
g(z) = z + \frac{a_2}{2} z^3 + \cdots.
\]

By (1), the conjugate map \( \psi_g = \iota \circ g \circ \iota \) has the Laurent expansion

\[
\psi_g(w) = w - \frac{a_2}{2w} + \cdots.
\]

Applying the inequality (2) to \( \psi_g \), we obtain \( |a_2| \leq 2 \).

By Theorem 5.10, the equality \( |a_2| = 2 \) occurs if and only if the complement of \( \psi_g(\mathbb{C} \setminus \mathbb{D}) \) is a straight segment of the form \([-2e^{i\theta}, 2e^{i\theta}]\). It easily follows from (4) that this is equivalent to \( f(\mathbb{D}) = \mathbb{C} \setminus [e^{-2i\theta}/4, \infty)\). \( \square \)

**Remark 5.13.** More generally, Bieberbach conjectured in 1916 that whenever \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S \), the inequality \( |a_k| \leq k \) must hold for all \( k \). This conjecture was proved by de Branges in 1984.

As an application of the Bieberbach inequality, we prove the following important result:

**Theorem 5.14** (Koebe 1/4-Theorem, 1914). If \( f \in S \), then \( f(\mathbb{D}) \supset \mathbb{D}(0, 1/4) \).

The example of Koebe function \( f(z) = z/(1 - z)^2 \) shows that the bound \( 1/4 \) is optimal (compare Example 5.8).

PROOF. Take an arbitrary point \( p \) outside \( f(\mathbb{D}) \). Let \( g \in \text{Aut}(\mathbb{C}) \) be the unique M"obius map which fixes the origin, has derivative 1 there, and sends \( p \) to \( \infty \):

\[
g(z) = \frac{pz}{p - z}.
\]
The composition \( g \circ f \) is easily seen to be in \( S \). If \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \), then

\[
(g \circ f)(z) = \frac{pf(z)}{p - f(z)} = \frac{p(z + a_2 z^2 + \cdots)}{z + a_2 z^2 + \cdots} = \frac{z + a_2 z^2 + \cdots}{1 - \left( \frac{1}{p} + \frac{a_2}{p} \right) z^2 + \cdots} = (z + a_2 z^2 + \cdots)(1 + \frac{1}{p}z + \cdots) = z + \left( a_2 + \frac{1}{p} \right) z^2 + \cdots.
\]

By the Bieberbach inequality (Theorem 5.12) applied to \( f \) and \( g \circ f \), both inequalities \(|a_2| \leq 2\) and \(|a_2 + 1/p| \leq 2\) must hold. Hence

\[
\frac{1}{|p|} \leq \left| a_2 + \frac{1}{p} \right| + |a_2| \leq 4,
\]

or \(|p| \geq 1/4\), as required. \( \square \)

**Problems**

1. Find an element of \( \text{Aut}(\hat{\mathbb{C}}) \) which maps the upper half-disk
   
   \[ U = \{ z = x + iy \in \mathbb{D} : y > 0 \} \]
   
   onto the first quadrant \( \{ z = x + iy \in \mathbb{C} : x > 0, y > 0 \} \). Use this map to find the explicit formula for a biholomorphism \( f : U \to \mathbb{D} \).

2. Let \( f : \mathbb{D} \to U \) be a biholomorphism, with \( f(0) = 0 \). If \( g : \mathbb{D} \to U \) is any holomorphic function with \( g(0) = 0 \), show that \( g(\mathbb{D}(0,r)) \subset f(\mathbb{D}(0,r)) \) for all \( 0 < r < 1 \). This is known as the subordination principle.

3. Let \( U \subset \mathbb{C} \) be simply connected, \( p \in U \), and \( f : U \to \mathbb{D} \) be the unique biholomorphism with \( f(p) = 0 \), \( f'(p) > 0 \). If \( g : U \to \mathbb{D} \) is any holomorphic function, show that \( |g'(p)| \leq f'(p) \), with equality if and only if \( g = \lambda f \) for some constant \( \lambda \) with \( |\lambda| = 1 \).

4. Let \( n \geq 2 \) be an integer and \( \omega = e^{2\pi i/n} \). Suppose \( U \subset \mathbb{C} \) is a simply-connected domain such that \( z \in U \) if and only if \( \omega z \in U \) (thus, \( U \) has a rotational symmetry by the angle \( 2\pi/n \)). Consider a Riemann map \( f : \mathbb{D} \to U \) with \( f(0) = 0 \). If \( f(z) = \sum_{k=1}^{\infty} a_k z^k \), prove that
   
   \[ a_k = 0 \quad \text{if } k \neq 1 \text{mod } n. \]
   
   As an example, if \( U \) has a 180° symmetry about the origin, its Riemann map can contain only the odd powers of \( z \). (Hint: Show that \( f(\omega z) = \omega f(z) \) for all \( z \in \mathbb{D} \).)

5. Let \( U \subset \mathbb{C} \) be a simply-connected domain and \( f : U \to U \) be a holomorphic function.
   
   (i) Show that \( f \) has at most one fixed point in \( U \) unless it is the identity map.

   (ii) If \( p = f(p) \), show that \( |f'(p)| \leq 1 \).

   (iii) If \( p = f(p) \) and \( |f'(p)| = 1 \), show that \( f \) is an automorphism \( U \to U \).
(iv) If \( p = f(p) \) and \( f'(p) = 1 \), show that \( f \) is the identity map.

(6) Let \( P(z) = z + a_2 z^2 + \cdots + a_n z^n \) be a polynomial in \( S \). Prove that
\[
|a_n| \leq \frac{1}{n}.
\]
For each \( n \geq 2 \), show by an example that this is the best upper bound. (Hint: The derivative of \( P \) does not vanish in \( \mathbb{D} \).)

(7) Suppose \( f \in S \), and for \( 0 < r < 1 \) let \( A(r) \) denote the Euclidean area of the domain \( f(\mathbb{D}(0, r)) \). Show that
\[
A(r) \geq \pi r^2.
\]
(Hint: Express \( A(r) \) either by Green’s Theorem as
\[
A(r) = \frac{1}{2i} \int_{f(\mathbb{D}(0, r))} \overline{w} \, dw = \frac{r}{2} \int_0^{2\pi} f(re^{it}) f'(re^{it}) e^{it} \, dt,
\]
or by the change of variable formula as the double integral
\[
A(r) = \iint_{f(\mathbb{D}(0, r))} \, dx \, dy = \int_0^r \int_0^{2\pi} \rho |f'(\rho e^{it})|^2 \, dt \, d\rho.
\]
In either case, substitute the power series of \( f \) and estimate the integral from below.)

(8) The function \( f : \mathbb{D} \to \mathbb{C} \) defined by \( f(z) = (e^{5z} - 1)/5 \) is holomorphic, with \( f(0) = 0 \) and \( f'(0) = 1 \). However, \( f \) does not take the value \(-1/5\). Why doesn’t this contradict Koebe’s \( 1/4 \)-Theorem?

(9) Suppose \( U \subseteq \mathbb{C} \) is a simply connected domain and \( p \in U \). Let \( f : \mathbb{D} \to U \) be any biholomorphism which sends \( 0 \) to \( p \). Define the **conformal radius** of \( U \) at \( p \) by \( \rho(U, p) = |f'(0)| > 0 \).

(i) Verify that the definition of \( \rho(U, p) \) does not depend on the particular choice of \( f \).

(ii) If \( V \subseteq \mathbb{C} \) is another simply connected domain and \( \varphi : U \to V \) is holomorphic with \( \varphi(p) = q \), show that
\[
|\varphi'(p)| \leq \frac{\rho(V, q)}{\rho(U, p)}
\]
and that equality holds if and only if \( \varphi \) is a biholomorphism.

(iii) Determine \( \rho(\mathbb{D}, z) \) for \( z \in \mathbb{D} \).

(iv) Let \( r(U, p) \) denote the Euclidean **inner radius** of \( U \) at \( p \), i.e., the radius of the largest open disk centered at \( p \) and contained in \( U \). Show that
\[
1 \leq \frac{\rho(U, p)}{r(U, p)} \leq 4.
\]

(10) (Bernstein) Consider a polynomial \( P(z) = a_d z^d + \cdots + a_1 z + a_0 \) with \( a_d \neq 0 \). Set \( M = \sup_{z \in [-1,1]} |P(z)| \).

(i) Show that
\[
|P(z)| \leq M (a + b)^d \quad \text{for all} \quad z \in \mathbb{C} \setminus [-1, 1],
\]
where \( a \) and \( b \) are the semi-axes of the ellipse through \( z \) with foci \( \pm 1 \).

(ii) Show that
\[
2^{-d} |a_d| \leq M.
\]
(Hint: For (i), note that the rational function $\zeta \mapsto \frac{1}{2}(\zeta + \zeta^{-1})$ maps the region $\{\zeta : |\zeta| > 1\}$ biholomorphically onto $\mathbb{C} \setminus [-1, 1]$, sending the family of circles $|\zeta| = \text{const.}$ to the family of ellipses with foci $\pm 1$. Apply problem 56 in chapter 1 to the function $f(\zeta) = \zeta^{-d} P\left(\frac{1}{2}(\zeta + \zeta^{-1})\right)$. For (ii), let $z \to \infty$ in (i).)

(11) Suppose $f \in \mathcal{O}(\mathbb{D})$, $0 < r < 1$, and $f'(z) \neq 0$ if $|z| = r$. Let $\Gamma_r$ be the image of the circle $|z| = r$ under $f$. Show that the distance from the origin to the line tangent to $\Gamma_r$ at $f(z)$ is given by
\[
\frac{\text{Re}\left(zf'(z)f(z)\right)}{|zf''(z)|}.
\]

(12) Under the assumptions of the previous problem, suppose $\Gamma_r$ has no self-intersection. Show that $\Gamma_r$ is a strictly convex loop if and only if
\[
\text{Re}\left(\frac{zf''(z)}{f'(z)}\right) > 1 \quad \text{whenever } |z| = r.
\]
CHAPTER 6

Harmonic Functions

This chapter studies basic properties of harmonic functions in planar domains. Although the theory can be fully developed without any appeal to complex analysis, the beautiful interplay between holomorphic and harmonic functions in dimension two provides a quick route to the essential results that are discussed here.

§6.1. What is a harmonic function?

Recall that the Laplace operator (or the Laplacian) \( \Delta \) acting on \( C^2 \) complex-valued functions is defined by

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]

It is well known that a \( C^2 \) function \( \varphi \) has equal mixed partial derivatives:

\[
\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}.
\]

Using this, it is easy to verify the following expressions for \( \Delta \) in terms of the complex differential operators \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \bar{z}} \):

\[
(1) \quad \Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}
\]

(compare problem [4] in Chapter [1]). For example,

\[
4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

\[
= \frac{\partial^2}{\partial x^2} + i \frac{\partial^2}{\partial x \partial y} - i \frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial y^2}
\]

\[
= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]

As usual, \( U \) will denote a non-empty open subset of \( \mathbb{C} \) unless otherwise stated.

Definition 6.1. A \( C^2 \) function \( \varphi : U \to \mathbb{C} \) is called harmonic if it satisfies the Laplace equation \( \Delta \varphi = 0 \) throughout \( U \).
Linearity of $\Delta$ shows that the space of all harmonic functions in $U$ is a vector space: If $\varphi$ and $\psi$ are harmonic in $U$, so is $\alpha \varphi + \beta \psi$ for every $\alpha, \beta \in \mathbb{C}$.

**Theorem 6.2.** The following conditions on a $C^2$ function $\varphi : U \to \mathbb{C}$ are equivalent:

(i) $\varphi$ is harmonic.

(ii) $\varphi_z = \frac{1}{2}(\varphi_x - i \varphi_y)$ is holomorphic.

(iii) $\varphi_{\bar{z}} = \frac{1}{2}(\varphi_x + i \varphi_y)$ is anti-holomorphic.

Recall that $f : U \to \mathbb{C}$ is anti-holomorphic if $f_z = 0$ in $U$; equivalently, if $\bar{f} \in O(U)$ (compare problem 5 in chapter 1).

**Proof.** By (i), $\Delta \varphi = 0$ if and only if $\varphi_{z \bar{z}} = (\varphi_z)_{\bar{z}} = 0$. By the complex Cauchy-Riemann equation (Corollary 1.13), this happens precisely when $\varphi_z$ is holomorphic. Thus (i) and (ii) are equivalent. Similarly, $\varphi$ is harmonic if and only if $\varphi_{z \bar{z}} = (\varphi_{\bar{z}})_z = 0$, which is the case precisely when $\varphi_{\bar{z}}$ is anti-holomorphic. \qed

**Example 6.3.** The function $\varphi(x, y) = e^x \cos y$ is harmonic as a simple computation shows. We have

$$\varphi_z = \frac{1}{2}(\varphi_x - i \varphi_y) = \frac{1}{2}(e^x \cos y + i e^x \sin y) = \frac{1}{2}e^z,$$

which is holomorphic.

**Corollary 6.4.** Suppose $\varphi : U \to \mathbb{C}$ is $C^2$.

(i) $\varphi$ is harmonic if and only if $\bar{\varphi}$ is harmonic.

(ii) $\varphi$ is harmonic if and only if $\text{Re} \, \varphi$ and $\text{Im} \, \varphi$ are harmonic.

(iii) If $\varphi$ is holomorphic or anti-holomorphic, then $\varphi$ is harmonic.

(iv) If $f : V \to U$ is holomorphic or anti-holomorphic and $\varphi$ is harmonic, then $\varphi \circ f$ is harmonic in $V$.

**Proof.** By Theorem 6.2

$$\varphi \text{ is harmonic } \iff \varphi_z \text{ is holomorphic}$$

$$\iff \varphi_{z\bar{z}} = \bar{\varphi}_{\bar{z}} \text{ is anti-holomorphic}$$

$$\iff \bar{\varphi} \text{ is harmonic}.$$

This proves (i).

Part (ii) follows from (i), the linearity of $\Delta$, and the relations $\text{Re} \, \varphi = (\varphi + \bar{\varphi})/2$, $\text{Im} \, \varphi = (\varphi - \bar{\varphi})/(2i)$.

If $\varphi$ is holomorphic, so is $\varphi_z = \varphi'$, hence $\varphi$ is harmonic by Theorem 6.2. If $\varphi$ is anti-holomorphic, then $\bar{\varphi}$ is holomorphic, so $\bar{\varphi}$ is harmonic, so $\varphi$ is harmonic by part (i). This proves (iii).
To show (iv), first assume $f$ is holomorphic. By the chain rule,

$$(\varphi \circ f)_z = (\varphi_z \circ f)f_z + (\varphi_{\bar{z}} \circ f)\bar{f}_z = (\varphi_z \circ f)f_z,$$

where we have used $\bar{f}_z = 0$ since $f$ is holomorphic. It follows that $(\varphi \circ f)_z$ is holomorphic because $f$, $f_z$, and $\varphi_z$ are. By Theorem 6.2, $\varphi \circ f$ is harmonic.

The case where $f$ is anti-holomorphic is similar. By the chain rule,

$$(\varphi \circ f)_z = (\varphi_z \circ f)f_z + (\varphi_{\bar{z}} \circ f)\bar{f}_z = (\varphi_z \circ f)f_z,$$

where we have used $\bar{f}_z = 0$ since $f$ is anti-holomorphic. It follows that $(\varphi \circ f)_z$ is anti-holomorphic because $f$, $f_z$, and $\varphi_z$ are. By Theorem 6.2, $\varphi \circ f$ is harmonic.

\[\square\]

**Corollary 6.5.** The real and imaginary parts of a holomorphic function are harmonic.

**Example 6.6.** The function

$$\varphi(z) = \begin{cases} \text{Im} \left( \frac{1}{z^2} \right) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is harmonic in the punctured plane $\mathbb{C}^*$ since it is the imaginary part of the holomorphic function $1/z^2$ there. The restriction of $\varphi$ to the coordinate axes $x = 0$ and $y = 0$ is identically zero, hence $\varphi_{xx}$ and $\varphi_{yy}$ are both zero at the origin. It follows that $\Delta \varphi = 0$ everywhere in the plane.

However, $\varphi$ is not harmonic in the plane because it is not even continuous at the origin.

The question arises as to whether every real-valued harmonic function in a domain is the real part of a holomorphic function in that domain. In general, the answer is negative, as the following example shows.

**Example 6.7.** The function $\varphi : \mathbb{C}^* \to \mathbb{R}$ defined by $\varphi(z) = \log |z|$ is harmonic:

$$\Delta \varphi = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log |z| = 2 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log(z\bar{z}) = 2 \frac{\partial}{\partial z} \left( \frac{1}{z} \right) = 0.$$

However, $\varphi$ is not the real part of any holomorphic function in $\mathbb{C}^*$. In fact, assuming there is an $f \in \mathcal{O}(\mathbb{C}^*)$ with $\text{Re}(f(z)) = \log |z|$, the function $g = \exp(f) \in \mathcal{O}(\mathbb{C}^*)$ satisfies $|g(z)| = |z|$ for all $z \neq 0$. The function $g(z)/z$ is holomorphic in $\mathbb{C}^*$ and has absolute value 1 everywhere, hence it must be constant by the Open Mapping Theorem, that is, $g(z) = \lambda z$ for some constant $\lambda$ with $|\lambda| = 1$. In particular, $g$ has a removable singularity at 0, with $g(0) = 0$. This implies $f$ has a removable singularity at 0 as well because a pole or an essential singularity at 0 would mean any small neighborhood $D$ of 0 maps under $f$ to an open set containing arbitrarily large positive numbers, hence $g(D)$ would not be bounded (compare problem 38 in chapter 1). But 0 cannot be removable for $f$ since $\text{Re}(f(z)) \to -\infty$ as $z \to 0$. The contradiction shows there is no such $f$.

However, it is not hard to show that every real-valued harmonic function is *locally* the real part of a holomorphic function. In fact, we can prove this in every simply connected domain:
**Theorem 6.8.** Suppose $U \subset \mathbb{C}$ is simply connected and $\varphi : U \rightarrow \mathbb{R}$ is harmonic. Then there exists an $f \in \mathcal{O}(U)$ such that $\text{Re}(f) = \varphi$ everywhere in $U$. Such $f$ is a primitive of the holomorphic function $2\varphi_z$ and is unique up to the addition of a purely imaginary constant.

**Proof.** By Theorem ?? the holomorphic function $2\varphi_z$ has a primitive $g \in \mathcal{O}(U)$. From $(g - 2\varphi)_z = g' - 2\varphi_z = 0$ we see that $g - 2\varphi$ is anti-holomorphic. Taking complex conjugates and noting that $\varphi$ is real-valued, it follows that $\bar{g} - 2\varphi \in \mathcal{O}(U)$. Thus, the sum $g + \bar{g} - 2\varphi$ of these two holomorphic functions must be holomorphic in $U$. Since this function is real-valued, it must be a constant $c$ by the Open Mapping Theorem. It follows that $\varphi$ is the real part of the holomorphic function $f = g - c/2$. If $\hat{f} \in \mathcal{O}(U)$ has its real part equal to $\varphi$, then the holomorphic function $f - \hat{f}$ has vanishing real part, hence it will be a purely imaginary constant, again by the Open Mapping Theorem. \[\square\]

The definition of a harmonic function assumed only $C^2$ smoothness. But since by Theorem 6.8 every real-valued harmonic function is locally the real part of a holomorphic function, it now follows that

**Corollary 6.9.** Harmonic functions are $C^\infty$ smooth.

Even more is true: harmonic functions are “real analytic” in the sense that they can be locally expanded into convergent power series in $x$ and $y$, or equivalently in $z$ and $\bar{z}$. We will discuss this in chapter 7.

**Corollary 6.10** (Identity Theorem for harmonic functions). Suppose $U \subset \mathbb{C}$ is a domain and $\varphi, \psi : U \rightarrow \mathbb{C}$ are harmonic. If $\varphi = \psi$ in a non-empty open subset of $U$, then $\varphi = \psi$ in $U$.

**Proof.** It suffices to consider the case where $\varphi$ is real-valued and $\psi = 0$; the general case will then follow by considering the real and imaginary parts of $\varphi - \psi$. Let $V \neq \emptyset$ be the maximal open subset of $U$ in which $\varphi = 0$. If $V \neq U$, choose a point $p \in \partial V \cap U$ and a disk $D(p, r) \subset U$. Since disks are simply connected, Theorem 6.8 gives a function $f \in \mathcal{O}(D(p, r))$ such that $\text{Re}(f) = \varphi$. Since $f(z)$ is purely imaginary for all $z$ in the open set $D(p, r) \cap V$, the open mapping theorem shows that $f$ must be constant in this set, hence in the disk $D(p, r)$. This shows $\varphi$ vanishes in the open set $D(p, r) \cup V$, which contradicts the maximality of $V$. Thus $V = U$ and $\varphi$ vanishes identically in $U$. \[\square\]

**Example 6.11.** Unlike the case of holomorphic functions, the above theorem is no longer true under the weaker assumption that the harmonic functions match along a set with accumulation points. For example, the harmonic function $z \mapsto \text{Re}(z)$ vanishes along the imaginary axis without vanishing identically.
Definition 6.12. We say that a continuous function \( \varphi : U \to \mathbb{C} \) has the **mean value property** if

\[
\varphi(p) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(p + re^{it}) \, dt \quad \text{whenever } \overline{D}(p, r) \subset U.
\]

If for every \( p \in U \) there is a small enough \( \delta > 0 \) such that (2) holds whenever \( 0 < r < \delta \), we say that \( \varphi \) has the **local mean value property**.

Since the averaging condition (2) is linear in \( \varphi \), it is clear that \( \varphi \) has the (local) mean value property if and only if \( \text{Re}(\varphi) \) and \( \text{Im}(\varphi) \) do.

The mean value property says that the average value of \( \varphi \) over the boundary of any closed disk contained in \( U \) equals the value of \( \varphi \) at the center, irrespective of what the radius of the disk is. The local mean value property says that the same holds provided that the radius of the disk is sufficiently small. The latter is apparently a weaker condition than the former, but we will see in Theorem 6.26 that they are actually equivalent to the condition of \( \varphi \) being harmonic. One direction of this statement is easy to prove:

**Theorem 6.13.** Harmonic functions have the mean value property.

**Proof.** Suppose \( \varphi \) is harmonic in an open set \( U \). It suffices to assume \( \varphi \) is real-valued; the general case follows by considering the real and imaginary parts. Let \( \overline{D}(p, r) \subset U \). By Theorem 6.8, there is an \( f \in \mathcal{O}(\overline{D}(p, r)) \) such that \( \text{Re}(f) = \varphi \). Use the parametrization \( \gamma(t) = p + re^{it}, t \in [0, 2\pi] \), for the oriented circle \( T(p, r) \) and apply Cauchy’s Integral Formula (Theorem 1.36) to obtain

\[
f(p) = \frac{1}{2\pi i} \int_{T(p, r)} \frac{f(\zeta)}{\zeta - p} \, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(p + re^{it})}{re^{it}} \, ire^{it} \, dt = \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{it}) \, dt.
\]

Now (2) follows by taking the real part of each side of this equation. \( \square \)

**Corollary 6.14** (Maximum Principle). Suppose \( U \subset \mathbb{C} \) is a bounded domain, \( \varphi \) is continuous on \( \overline{U} \) and has the local mean value property in \( U \) (in particular, this holds if \( \varphi \) is harmonic). Then

\[
|\varphi(z)| \leq \sup_{\zeta \in \partial U} |\varphi(\zeta)| \quad \text{for all } z \in U.
\]

If \( \varphi \) is real-valued, then

\[
\inf_{\zeta \in \partial U} \varphi(\zeta) \leq \varphi(z) \leq \sup_{\zeta \in \partial U} \varphi(\zeta) \quad \text{for all } z \in U.
\]

In either case, if the equality occurs at some \( z \in U \), then \( \varphi \) is constant.
PROOF. First consider the case \( \varphi \) is real-valued. Suppose \( \varphi(z) \geq \sup_{\zeta \in \partial U} \varphi(\zeta) \) for some \( z \in U \). Then the supremum of \( \varphi \) on the compact set \( \overline{U} \) occurs at some point \( z_0 \in U \). Consider the non-empty set \( E = \{ z \in U : \varphi(z) = \varphi(z_0) \} \), which is closed since \( \varphi \) is continuous. If \( z \in E \), the local mean value property shows that there is a \( \delta > 0 \) such that

\[
\varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + re^{it}) \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(z_0) \, dt = \varphi(z_0) = \varphi(z)
\]

for every \( 0 < r < \delta \). It follows that \( \varphi(z + re^{it}) = \varphi(z_0) \) whenever \( 0 < r < \delta \) and \( 0 \leq t \leq 2\pi \), which means \( \mathbb{D}(z, \delta) \subseteq E \). Thus \( E \) is open. Since \( U \) is connected, we must have \( E = U \), and \( \varphi \) will be constant in \( U \). The left inequality in (4) follows by applying the above argument to \( -\varphi \).

Now suppose \( \varphi \) is complex-valued and \( |\varphi(z)| \geq \sup_{\zeta \in \partial U} |\varphi(\zeta)| \) for some \( z \in U \). Then the supremum of \( |\varphi| \) on \( \overline{U} \) occurs at some point \( z_0 \in U \). If \( \varphi(z_0) = 0 \), then \( \varphi \) vanishes identically and there is nothing to prove. If \( \varphi(z_0) \neq 0 \), consider the continuous function \( \psi(z) = \varphi(z)/\varphi(z_0) \) which has the local mean value property. Then \( |\psi| \leq 1 \) and \( \psi(z_0) = 1 \). Hence Re(\( \psi \)) takes its maximum value at \( z_0 \). Since Re(\( \psi \)) has the local mean value property, by the first case treated above, Re(\( \psi \)) = 1 everywhere. Thus the image \( \psi(U) \) will be contained in the vertical line \( \text{Re}(z) = 1 \) as well as the closed unit disk \( \overline{\mathbb{D}} \). It follows that \( \psi \) must be identically 1, hence \( \varphi(z) = \varphi(z_0) \) for all \( z \in U \). \( \square \)

Remark 6.15. The argument showing \( \varphi \) is constant in the first case above is quite similar to the solution of the following classical problem: “Suppose 64 numbers are written in the squares of an \( 8 \times 8 \) chessboard in such a way that the number in each square is the average of the numbers in the neighboring squares. Show that all the numbers are equal.” In fact, starting with a square which has the largest assigned number and using the average property, we see that all its neighboring squares must have equal numbers in them. Now repeat the argument with one of these neighboring squares until all the chessboard is eventually covered.

§6.2. Poisson formula in a disk

We begin by a fundamental representation theorem for harmonic functions in the unit disk. It can be viewed as a generalization of the mean value property, or an analog of the Cauchy Integral Formula for harmonic functions.

Theorem 6.16 (Poisson Integral Formula, 18??). If \( \varphi \) is continuous on \( \overline{\mathbb{D}} \) and harmonic in \( \mathbb{D} \), then

\[
\varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \varphi(e^{it}) \, dt \quad z \in \mathbb{D}.
\]
Thus, the value of $\varphi$ at every point $z \in \mathbb{D}$ is still an average of the values of $\varphi$ on the unit circle, weighted by the factor $(1 - |z|^2)/|e^{it} - z|^2$.

**Proof.** Fix $p \in \mathbb{D}$ and consider the disk automorphism $z \mapsto w = \phi_{-p}(z) = (z + p)/(1 + \bar{p}z)$ which sends $0$ to $p$, with the inverse $w \mapsto z = \phi_p(w) = (w - p)/(1 - \bar{p}w)$. The function $\psi = \varphi \circ \phi_{-p}$ is continuous on $\mathbb{D}$ and harmonic in $\mathbb{D}$ (Corollary 6.4(iv)). Hence, by the mean value property,

$$
\varphi(p) = \psi(0) = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{it}) \, dt = \frac{1}{2\pi i} \int_{\mathbb{T}} \psi(z) \frac{dz}{z} = \frac{1}{2\pi i} \int_{\mathbb{T}} \psi(\phi_p(w)) \frac{\phi'(w)}{\phi_p(w)} \, dw
$$

But when $|w| = 1$,

$$
\frac{\phi'(w)}{\phi_p(w)} = \frac{1 - |p|^2}{|w - p|^2} = \frac{1 - |p|^2}{(w - p)(\bar{w} - \bar{p})w} = \frac{1 - |p|^2}{|w - p|^2w}.
$$

It follows that

$$
\varphi(p) = \frac{1}{2\pi i} \int_{\mathbb{T}} \varphi(w) \frac{1 - |p|^2}{|w - p|^2} \, dw = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |p|^2}{|e^{it} - p|^2} \psi(e^{it}) \, dt
$$

as required. \qed

In particular, when $\varphi$ is real-valued, the Poisson Integral Formula (5) allows us to find an explicit formula for a function $f \in \mathcal{O}(\mathbb{D})$ with $\text{Re}(f) = \varphi$, whose existence we have already shown in Theorem 6.8.

**Corollary 6.17.** If $\varphi$ is real-valued, continuous on $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$, then $\varphi$ is the real part of the holomorphic function

$$
f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \varphi(e^{it}) \, dt \quad z \in \mathbb{D}.
$$

**Proof.** Corollary [4.45](#) shows that $f \in \mathcal{O}(\mathbb{D})$. That $\text{Re}(f) = \varphi$ follows from (5) and the computation

$$
\text{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) = \text{Re}\left(\frac{(e^{it} + z)(e^{-it} - \bar{z})}{|e^{it} - z|^2}\right) = \frac{1 - |z|^2}{|e^{it} - z|^2}.
$$

\(\square\)
Definition 6.18. The Poisson kernel in the unit disk \( \mathbb{D} \) is defined by

\[
P(\zeta, z) = \text{Re} \left( \frac{\zeta + z}{\zeta - z} \right) = \frac{1 - |z|^2}{|\zeta - z|^2},
\]

where \( \zeta, z \) are complex numbers with \( |\zeta| = 1 \) and \( |z| < 1 \).

We often write \( \zeta = e^{it} \), where \( 0 \leq t \leq 2\pi \), in which case (7) takes the form

\[
P(e^{it}, z) = \text{Re} \left( \frac{e^{it} + z}{e^{it} - z} \right) = \frac{1 - |z|^2}{|e^{it} - z|^2}.
\]

If we use the polar form \( z = re^{i\theta} \), where \( 0 \leq r < 1 \) and \( 0 \leq \theta \leq 2\pi \), an easy computation shows that

\[
P(e^{it}, re^{i\theta}) = \frac{1 - r^2}{1 - 2r \cos(t - \theta) + r^2}.
\]

We may view the Poisson kernel \( P(e^{it}, z) \) either as a family of functions \( t \mapsto P(e^{it}, z) \) on the interval \([0, 2\pi]\) parametrized by the point \( z \in \mathbb{D} \), or as a family of functions \( z \mapsto P(e^{it}, z) \) in the unit disk parametrized by the point \( t \in [0, 2\pi] \).

Theorem 6.19 (Basic properties of the Poisson kernel).

(i) For each \( \zeta \in \mathbb{T} \), \( z \mapsto P(\zeta, z) \) is harmonic in \( \mathbb{D} \).
The graph of the Poisson kernel $P(\zeta_0, z)$ over the unit disk $|z| < 1$. Notice that $P(\zeta_0, z) \to 0$ as $z$ tends to any point of the unit circle other than $\zeta_0$ where there is a spike. On the other hand, Theorem 6.20 shows that $P(\zeta_0, z)$ can get arbitrarily close to any number in $[0, +\infty]$ as $z \to \zeta_0$.

(ii) $P(\zeta, z) > 0$ for every $\zeta \in \mathbb{T}$ and $z \in \mathbb{D}$. Moreover,

$$\max_{\zeta \in \mathbb{T}} P(\zeta, z) = \frac{1 + |z|}{1 - |z|} \quad \text{and} \quad \min_{\zeta \in \mathbb{T}} P(\zeta, z) = \frac{1 - |z|}{1 + |z|}.$$ 

(iii) $\frac{1}{2\pi} \int_0^{2\pi} P(e^{it}, z) \, dt = 1$ for all $z \in \mathbb{D}$.

(iv) For every $\zeta_0 \in \mathbb{T}$ and every open interval $I \subseteq \mathbb{T}$ centered at $\zeta_0$, $P(\zeta, z) \to 0$ uniformly in $\zeta \in \mathbb{T} \setminus I$ as $z \to \zeta_0$.

As $t \mapsto P(e^{it}, z)$ is positive with average value 1 on $[0, 2\pi]$, property (iv) implies that as $z$ tends to $e^{it_0} \in \mathbb{T}$, the graph of $t \mapsto P(e^{it}, z)$ appears as a high spike concentrated near the point $t_0$ (compare Fig. 1).

**Proof.** (i) follows from Corollary 6.5 since for each fixed $\zeta \in \mathbb{T}$, $z \mapsto P(\zeta, z)$ is the real part of the holomorphic function $z \mapsto (\zeta + z)/(\zeta - z)$ in $\mathbb{D}$.

Positivity of $P(\zeta, z)$ follows at once from (7). To verify the formulas for the extreme values of the Poisson kernel, it suffices to consider the case $z \neq 0$ since $P(\zeta, 0) = 1$ for all $\zeta$. By (7), $\zeta \mapsto P(\zeta, z)$ takes on its maximum value when $|\zeta - z|$ reaches its minimum value of $1 - |z|$ at $\zeta = z/|z|$. Similarly, $\zeta \mapsto P(\zeta, z)$ takes on its minimum value when $|\zeta - z|$ reaches its maximum value of $1 + |z|$ at $\zeta = -z/|z|$ (compare Fig. 1). Substituting these values into the formula for $P(\zeta, z)$ proves (ii).
Part (iii) follows from the Poisson Integral Formula \( (5) \) applied to the constant function \( \varphi = 1 \).

Property (iv) is evident since as \( z \to \zeta_0 \), \( 1 - |z|^2 \to 0 \) but \( |\zeta - z|^2 \) is uniformly bounded away from 0 on \( \mathbb{T} \setminus I \). More precisely, let \( \ell > 0 \) be the distance between \( \zeta_0 \) and the endpoints of the interval \( I \). For a given \( \varepsilon > 0 \), let \( \delta = \min\{\varepsilon, \ell/2\} \). Then, if \( z \in \mathbb{D}(\zeta_0, \delta) \cap \mathbb{D} \) and \( \zeta \in \mathbb{T} \setminus I \),

\[
1 - |z|^2 = (1 + |z|)(1 - |z|) < 2\delta \leq 2\varepsilon
\]

while

\[
|\zeta - z| \geq |\zeta - \zeta_0| - |z - \zeta_0| > \ell - \delta \geq \frac{\ell}{2}.
\]

Hence

\[
P(\zeta, z) = \frac{1 - |z|^2}{|\zeta - z|^2} < \frac{8\varepsilon}{\ell^2}.
\]

Part (iv) of the above theorem shows that for every \( \zeta \in \mathbb{T} \setminus \{\zeta_0\} \), \( P(\zeta, z) \to 0 \) as \( z \to \zeta_0 \). One naturally asks what happens to \( P(\zeta_0, z) \) as \( z \to \zeta_0 \). Here is the answer:

**Theorem 6.20.** For every \( \zeta_0 \in \mathbb{T} \) and every \( 0 \leq \alpha \leq +\infty \) there is a sequence \( z_k \to \zeta_0 \) such that \( P(\zeta_0, z_k) \to \alpha \) as \( k \to \infty \).

**Proof.** The formula \( P(\zeta_0, z) = \text{Re}(\frac{(\zeta_0 + z)}{(\zeta_0 - z)}) \) provides the easiest way to see this. In fact, the Möbius map \( w = \frac{\zeta_0 + z}{\zeta_0 - z} \) maps \( \mathbb{D} \) biholomorphically
to the right half-plane $\text{Re}(w) > 0$, sending $\zeta_0$ to $\infty$, $-\zeta_0$ to 0 and 0 to 1. The Poisson kernel $P(\zeta_0, z)$ then becomes $\text{Re}(w)$. The family of vertical lines in the right half-plane pull back under $z = w^{-1} = \zeta_0(w-1)/(w+1)$ to the family of circles in $\mathbb{D}$ that are tangent to $\mathbb{T}$ at $\zeta_0$ (see Fig. 3). If $z \to \zeta_0$ along the pull-back of the vertical line $\text{Re}(w) = \alpha \in (0, +\infty)$, it follows that $P(\zeta_0, z) \to \alpha$. On the other hand, the family of horizontal lines in the right half-plane pull back to the family of circular arcs in $\mathbb{D}$ that meet $\mathbb{T}$ orthogonally at $0$. If $z \to \zeta_0$ along such circular arcs, it follows that $P(\zeta_0, z) \to +\infty$. Finally, any sequence $\{w_k\}$ in the right half-plane with $\text{Re}(w_k) \to 0$ and $|\text{Im}(w_k)| \to +\infty$ pulls back to a sequence $\{z_k\}$ in $\mathbb{D}$ which tends to $\zeta_0$ and has the property $P(\zeta_0, z_k) \to 0$. □

**Definition 6.21.** The Poisson integral of $\varphi \in L^1(\mathbb{T})$ is defined by

$$P[\varphi](z) = \frac{1}{2\pi} \int_0^{2\pi} P(e^{it}, z) \varphi(e^{it}) \, dt \quad z \in \mathbb{D}.$$ 

By (9), for $z = re^{i\theta}$ this can be written more explicitly as

$$P[\varphi](re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} \varphi(e^{it}) \, dt.$$ 

**Theorem 6.22** (Poisson integrals are harmonic). For any $\varphi \in L^1(\mathbb{T})$, the Poisson integral $P[\varphi]$ is harmonic in $\mathbb{D}$.

**Proof.** By Corollary [1.45], the function

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \varphi(e^{it}) \, dt$$

is holomorphic in $\mathbb{D}$. When $\varphi$ is real-valued, it follows that

$$P[\varphi](z) = \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left( \frac{e^{it} + z}{e^{it} - z} \right) \varphi(e^{it}) \, dt = \text{Re}(f(z)),$$

which is harmonic by Corollary 6.5. For a complex-valued $\varphi$, use this, Corollary 6.4(ii) and the decomposition

$$P[\varphi] = P[\text{Re} \varphi] + i \, P[\text{Im} \varphi],$$

to conclude that $P[\varphi]$ is harmonic. □

**Remark 6.23.** Alternatively, we could differentiate under the integral sign to obtain

$$\Delta P[\varphi](z) = \int_0^{2\pi} \Delta P(e^{it}, z) \varphi(e^{it}) \, dt$$

which is zero since by Theorem 6.19(i) the Poisson kernel itself is harmonic.
We now consider the question of the boundary values of Poisson integrals.

**Theorem 6.24.** If \( \varphi \in L^1(\mathbb{T}) \) is continuous at \( \zeta_0 = e^{i\theta_0} \), then

\[
\lim_{z \to \zeta_0} P[\varphi](z) = \varphi(\zeta_0).
\]

**Proof.** It suffices to consider the case \( \varphi(\zeta_0) = 0 \); the general case follows from this by considering the function \( \varphi - \varphi(\zeta_0) \). Fix \( \varepsilon > 0 \) and let \( I \subseteq \mathbb{T} \) be an open interval centered at \( \zeta_0 \) such that \( |\varphi(\zeta)| < \varepsilon \) whenever \( \zeta \in I \). Let \( J = \mathbb{T} \setminus I \). Define

\[ \varphi_I = \varphi \chi_I \quad \text{and} \quad \varphi_J = \varphi \chi_J, \]

where \( \chi_I, \chi_J \) are the characteristic functions of \( I, J \), respectively. The decomposition \( \varphi = \varphi_I + \varphi_J \) together with linearity of \( P[\cdot] \) show that

\[
P[\varphi] = P[\varphi_I] + P[\varphi_J].
\]

By Theorem 6.19(iii),

\[
|P[\varphi_I](z)| \leq \frac{1}{2\pi} \int_I P(e^{it}, z)|\varphi(e^{it})| \, dt \leq \varepsilon \cdot \frac{1}{2\pi} \int_I P(e^{it}, z) \, dt \leq \varepsilon.
\]

By Theorem 6.19(iv), there is a \( \delta > 0 \) such that \( P(\zeta, z) < \varepsilon \) whenever \( z \in D(\zeta_0, \delta) \cap \mathbb{D} \) and \( \zeta \in J \). This gives

\[
|P[\varphi_J](z)| \leq \frac{1}{2\pi} \int_J P(e^{it}, z)|\varphi(e^{it})| \, dt \leq \varepsilon \|\varphi\|_1.
\]

Whenever \( z \in D(\zeta_0, \delta) \cap \mathbb{D} \) (as usual, \( \|\varphi\|_1 \) is the \( L^1 \)-norm of \( \varphi \) on the circle). Putting (10) and (11) together, we conclude that

\[
|P[\varphi](z)| \leq |P[\varphi_J](z)| + |P[\varphi_I](z)| \leq (1 + \|\varphi\|_1)\varepsilon
\]

whenever \( z \in D(\zeta_0, \delta) \cap \mathbb{D} \). \( \Box \)

**Corollary 6.25.** Every continuous function \( \varphi : \mathbb{T} \to \mathbb{C} \) has a unique continuous extension \( \hat{\varphi} : \overline{\mathbb{D}} \to \mathbb{C} \) which is harmonic in \( \mathbb{D} \). It is given by

\[
\hat{\varphi}(z) = \begin{cases} P[\varphi](z) & z \in \mathbb{D} \\ \varphi(z) & z \in \mathbb{T} \end{cases}
\]

This solves the so-called **Dirichlet Problem** in the disk, i.e., the problem of finding a harmonic function in \( \mathbb{D} \) with given continuous boundary values. The problem for more general domains is quite delicate. We will discuss aspects of it in chapter ??.

**Proof.** \( \hat{\varphi} \) is harmonic in \( \mathbb{D} \) by Theorem 6.22 and continuous on \( \overline{\mathbb{D}} \) by Theorem 6.24. If \( \Psi \) is another continuous extension of \( \varphi \) to \( \overline{\mathbb{D}} \) which is harmonic in \( \mathbb{D} \), then Theorem 6.16 shows that

\[
\Psi = P[\Psi|_{\mathbb{T}}] = P[\varphi] = \hat{\varphi}
\]
everywhere in \( \mathbb{D} \). \( \Box \)
Everything we have said so far about the Poisson kernel in $\mathbb{D}$ can be stated for an arbitrary disk $\mathbb{D}(p, R)$ in the plane. This is achieved most conveniently by the affine change of coordinates $z \mapsto (z - p)/R$ which maps $\mathbb{D}(p, R)$ biholomorphically onto $\mathbb{D}$. The Poisson integral of a function $\varphi \in L^1(\mathbb{T}(p, R))$ then takes the form

$$P[\varphi](z) = \frac{1}{2\pi} \int_0^{2\pi} P\left(e^{it}, \frac{z - p}{R}\right) \varphi(p + Re^{it}) \, dt$$

or

$$P[\varphi](z) = \frac{1}{2\pi} \int_0^{2\pi} P(Re^{it}, z - p) \varphi(p + Re^{it}) \, dt$$

or

$$(12) \quad P[\varphi](z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z - p|^2}{|Re^{it} - (z - p)|^2} \varphi(p + Re^{it}) \, dt \quad z \in \mathbb{D}(p, R).$$

**Theorem 6.26.** The following conditions on a continuous function $\varphi : U \to \mathbb{C}$ are equivalent:

(i) $\varphi$ is harmonic.

(ii) $\varphi$ has the mean value property.

(iii) $\varphi$ has the local mean value property.

**Proof.** The implication (i) $\implies$ (ii) is Theorem 6.13 and (ii) $\implies$ (iii) is trivial. To prove (iii) $\implies$ (i), suppose $\varphi$ has the local mean value property. Take any disk $\mathbb{D}(p, R)$ with $\mathbb{D}(p, R) \subset U$. By Corollary 6.25, the function $\psi = P[\varphi|_{\mathbb{T}(p, R)}]$ is harmonic in $\mathbb{D}(p, R)$ and extends continuously to $\varphi$ on the circle $\mathbb{T}(p, R)$. The difference $h = \varphi - \psi$ is continuous on $\overline{\mathbb{D}}(p, R)$, has the local mean value property in $\mathbb{D}(p, R)$, and $h = 0$ on $\mathbb{T}(p, R)$. By the Maximum Principle (Corollary 6.14), $h$ must vanish identically in $\mathbb{D}(p, R)$. Hence $\varphi = \psi$ is harmonic in this disk. Since $\mathbb{D}(p, R)$ was arbitrary, we conclude that $\varphi$ is harmonic in $U$. \(\square\)

**Problems**

(1) Verify the formula

$$\Delta(\varphi \circ f) = (\Delta \varphi) \circ f \cdot |f'|^2$$

if $\varphi$ is smooth and $f$ is holomorphic. Use this to find another proof that $\varphi \circ f$ is harmonic whenever $\varphi$ is.

(2) Show that

(i) If $f : U \to \mathbb{C}^*$ is holomorphic, then $\log |f|$ is harmonic in $U$.

(ii) If $f$ and $g$ are holomorphic in a domain $U$ and the product $f \cdot g : U \to \mathbb{C}$ is harmonic, then either $f$ or $g$ is constant in $U$.

(3) Prove the following alternative expression for the Poisson kernel:

$$P(e^{it}, re^{i\theta}) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik(t - \theta)} \quad 0 \leq r < 1.$$
where for fixed $r$ the series converges uniformly in $t, \theta$.

(4) Give two alternative proofs for $\frac{1}{2\pi} \int_0^{2\pi} P(e^{it}, z) \, dt = 1$ by

(i) Computing

$$\frac{1}{2\pi} \int_0^{2\pi} e^{it} + z \, dt = \frac{1}{2\pi i} \int_\mathbb{C} \frac{\xi + z}{\xi - z} \frac{d\xi}{\xi}$$

using the Residue Theorem and taking the real parts.

(ii) Term-by-term integration of the series of the previous problem.

(5) Let $\psi : \mathbb{D} \to \mathbb{R}$ be harmonic with a continuous extension to $\mathbb{D}$. By (6),

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{e^{it} - z} \psi(e^{it}) \, dt = \frac{1}{2\pi i} \int_\mathbb{C} \frac{\psi(\zeta)}{\zeta - z} \, d\zeta,$$

is holomorphic in $\mathbb{D}$ with $f = \text{Re}(\psi)$. Compare this formula with Theorem 6.8 which asserts any such $f$ must be a primitive of $2\psi_z$. To this end, start with the representation (5) for $\psi$ and differentiate under the integral sign to show

$$2\psi_z(z) = \frac{1}{\pi i} \int_\mathbb{C} \frac{\psi(\zeta)}{(\zeta - z)^2} \, d\zeta.$$

A primitive $g \in \mathcal{O}(\mathbb{D})$ of $2\psi_z$ can be obtained by integration from 0 to $z$ radially. Use Fubini’s Theorem on interchanging the order of iterated integrals to prove

$$g(z) = \frac{1}{\pi i} \int_\mathbb{C} \frac{z}{\zeta - z} \psi(\zeta) \frac{d\zeta}{\zeta}.$$

Show that up to a constant, $g$ coincides with $f$: $f - g = \psi(0)$.

(6) Verify that the Laplace operator in polar coordinates $(r, \theta)$ has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

Show that every harmonic function $\psi$ in an annulus centered at the origin which depends on $r = |z|$ only must have the form

$$\psi(r) = \alpha \log r + \beta$$

for some constants $\alpha, \beta \in \mathbb{C}$.

(7) Let $\psi : \{z \in \mathbb{C} : a < |z| < b\} \to \mathbb{C}$ be harmonic and define

$$I(r) = \frac{1}{2\pi} \int_0^{2\pi} \psi(re^{it}) \, dt \quad a < r < b.$$ 

Show that $I(r) = \alpha \log r + \beta$ for some constants $\alpha, \beta \in \mathbb{C}$.

(8) Show that if $\psi : \mathbb{D}^*(p, r) \to \mathbb{C}$ is bounded and harmonic, then $\psi$ extends to a harmonic function in $\mathbb{D}(p, r)$.

(9) (Harnack’s Inequalities) Let $\psi$ be a positive harmonic function in a domain $U$ and $\mathbb{D}(p, R) \subset U$. Show that if $r < R$, then

$$\frac{R - r}{R + r} \psi(p) \leq \psi(z) \leq \frac{R + r}{R - r} \psi(p) \quad \text{for all } z \in \mathbb{D}(p, r).$$

(ii) Using (i), show that if a harmonic function $\psi : \mathbb{C} \to \mathbb{R}$ is bounded from above or below, it must be constant.
(iii) Give an alternative proof of (ii) by reducing to the case where \( \varphi \leq 0 \), finding an entire function \( f \) with \( \text{Re}(f) = \varphi \) and considering the entire function \( \exp(f) \).

(10) The map
\[
\zeta = f(z) = -\left( \frac{1 - z}{1 + z} \right)^2
\]
is the composition of the Möbius transformation \( w = i(1 - z)/(1 + z) \) followed by the squaring map \( \zeta = w^2 \).

(i) Use this to find the image of the unit disk \( \mathbb{D} \) under \( f \). In particular, describe (at least qualitatively) the images under \( f \) of the radial lines in \( \mathbb{D} \) coming out of the origin.

(ii) Define the harmonic function \( \varphi : \mathbb{D} \to \mathbb{R} \) by \( \varphi = \text{Im}(f) \). Show that for every \( p \neq -1 \) on the unit circle, \( \lim_{z \to p} \varphi(z) = 0 \). Show that the same is true of \( p = -1 \) provided that \( z \to -1 \) radially. So it seems \( \varphi \) has boundary value 0 without being 0 itself. Explain why this example does not contradict the Poisson Integral Formula.

(11) Let \( U \) be the open annulus \( \{ z \in \mathbb{C} : 0 < |z| < b \leq +\infty \} \) and \( \varphi : U \to \mathbb{R} \) be harmonic. Show that there is a constant \( \alpha \in \mathbb{R} \) and a holomorphic function \( f : U \to \mathbb{C} \) such that
\[
\text{Re}(f(z)) = \varphi(z) - \alpha \log |z| \quad z \in U.
\]

(12) (i) Verify that the unique harmonic function \( \varphi : \mathbb{D} \to \mathbb{C} \) whose boundary value is a trigonometric polynomial
\[
\varphi(e^{i\theta}) = a_0 + \sum_{k=1}^{n} [a_k \cos(k\theta) + b_k \sin(k\theta)]
\]
is given by the formula
\[
\varphi(re^{i\theta}) = a_0 + \sum_{k=1}^{n} r^k [a_k \cos(k\theta) + b_k \sin(k\theta)].
\]

(ii) Use (i) to find the solution of the Laplace equation \( \Delta \varphi = 0 \) in \( \mathbb{D} \) with the boundary value \( \varphi(e^{i\theta}) = \cos^2 \theta \). Compare this solution to the one given by the Poisson Integral Formula. Use this to find the value of the integral
\[
\int_{0}^{2\pi} \frac{\cos^2 t}{5 - 3\cos t} \, dt.
\]
CHAPTER 7

Analytic Continuation

The problem of analytic continuation can be roughly formulated as follows: “Given \( f \in \mathcal{O}(U) \), what are the possible holomorphic extensions of \( f \) to open sets containing \( U \)?” This chapter will address several formulations and refinements of this problem which often arise in practical situations.

§7.1. Regular and singular points

**Definition 7.1.** Let \( U \subset \mathbb{C} \) be open and \( f \in \mathcal{O}(U) \). A point \( q \in \partial U \) is called a **regular** point of \( f \) if there is a neighborhood \( V \) of \( q \) and a function \( g \in \mathcal{O}(V) \) such that \( f = g \) in \( U \cap V \). Otherwise, \( q \) is called a **singular** point of \( f \).

**Example 7.2.** Consider \( f(z) = \sum_{k=0}^{\infty} z^k = 1/(1-z) \) in \( \mathbb{D} \). Then \( q = 1 \) is a singular point of \( f \) while every \( q \in \mathbb{T} \setminus \{1\} \) is regular. Note, in particular, that \( q \) being regular does not imply the convergence of the power series \( \sum_{k=0}^{\infty} z^k \) at \( z = q \). In fact, \( \sum_{k=0}^{\infty} z^k \) diverges everywhere on \( \mathbb{T} \).

**Theorem 7.3.** Suppose the power series \( f(z) = \sum_{k=0}^{\infty} a_k (z-p)^k \) has radius of convergence \( 0 < R < +\infty \). Then \( f \) has at least one singular point on the circle \( \mathbb{T}(p, R) \).

**Proof.** Assume by way of contradiction that every \( q \in \mathbb{T}(p, R) \) is regular, and choose an open disk \( D_q \) centered at \( q \) and a function \( g_q \in \mathcal{O}(D_q) \) such that \( f = g_q \) in \( D_q \cap \mathbb{D}(p, R) \). Note that if \( D_q \cap D_{q'} \neq \emptyset \), then \( D_q \cap D_{q'} \cap \mathbb{D}(p, R) \neq \emptyset \) and \( g_q = f = g_{q'} \) in \( D_q \cap D_{q'} \cap \mathbb{D}(p, R) \), so \( g_q = g_{q'} \) in \( D_q \cap D_{q'} \) by the Identity Theorem. It follows that the function \( F \) defined by

\[
F(z) := \begin{cases} 
   f(z) & \text{if } z \in \mathbb{D}(p, R) \\
   g_q(z) & \text{if } z \in D_q 
\end{cases}
\]

is a well-defined holomorphic extension of \( f \) to the union \( U = \mathbb{D}(p, R) \cup \bigcup_{q \in \mathbb{T}(p, R)} D_q \). Since \( U \) is an open neighborhood of the compact set \( \overline{\mathbb{D}}(p, R) \), we can choose an \( \varepsilon > 0 \) small enough so that \( \mathbb{D}(p, R+\varepsilon) \subset U \). Thus, \( F \) can be represented by a convergent power series in \( \mathbb{D}(p, R+\varepsilon) \). But by Theorem 1.20 (iii), this power
series must coincide with $\sum_{k=0}^{\infty} a_k (z - p)^k$. This contradicts our assumption that the radius of convergence of this power series is $R$. □

It is clear from Definition [7.1] that the set of singular points of $f \in \mathcal{O}(U)$ is a closed subset of $\partial U$. It may happen to be the whole $\partial U$.

**Definition 7.4.** We say that $\partial U$ is the **natural boundary** of $f \in \mathcal{O}(U)$ if every point of $\partial U$ is a singular point of $f$.

**Theorem 7.5.** For every domain $U \subset \mathbb{C}$ there exists an $f \in \mathcal{O}(U)$ which has $\partial U$ as the natural boundary.

**Proof.** We can assume $U \neq \mathbb{C}$. Choose a discrete set $E \subset U$ which accumulates at every point of $\partial U$. By Theorem [3.38] there exists an $f \in \mathcal{O}(U)$ which has a simple zero at every point of $E$ and does not vanish anywhere else. Then $f$ has the desired property. In fact, if $f$ extends holomorphically to an open disk $D$ centered at some $q \in \partial U$, then $q$ is an accumulation point of the zeros of $f$ in $D$. It follows from the Identity Theorem that $f$ vanishes identically in $D$, hence in $U$, which is a contradiction. □

In particular, boundaries of disks can occur as natural boundaries. Examples of this case can also be constructed by carefully controlling the coefficients of power series.

**Example 7.6.** Consider $f \in \mathcal{O}(\mathbb{D})$ defined by $f(z) := \sum_{k=1}^{\infty} z^k$. Let $z = re^{it}$ and $t = 2\pi(a/b)$, where $a, b$ are integers with $b > 1$. Then

$$f(z) = \left( \sum_{k=1}^{b-1} + \sum_{k=b}^{\infty} \right) r^k e^{2\pi ikt} = \text{polynomial in } r + \sum_{k=b}^{\infty} r^k.$$

The second sum tends to $+\infty$ as $r \to 1$, hence $f(re^{it}) \to \infty$ as $r \to 1$ and $e^{it}$ is a singular point of $f$. Since the set of all such points $e^{it}$ is dense on $\mathbb{T}$ and since the set of singular points of $f$ is closed, we conclude that $\mathbb{T}$ is the natural boundary of $f$.

The preceding example is a special case of a more general phenomenon for power series with large gaps in their coefficients, which is stated in Theorem [7.9] below. We shall deduce this theorem from a stronger result that is the content of Theorem [7.8]

**Definition 7.7 (Ostrowski pairs).** A pair $(\{m_j\}, \{n_j\})$ of sequences of positive integers is called an **Ostrowski pair** if

- $m_1 < n_1 \leq m_2 < n_2 \leq \cdots \leq m_j < n_j \leq \cdots$
- there is a constant $\lambda > 1$ such that $n_j > \lambda m_j$ for all $j$. 

Theorem 7.8 (Ostrowski, 1921). Let \( f(z) = \sum_{k=0}^\infty a_k z^k \) have radius of convergence 1. Suppose there is an Ostrowski pair \( \{m_j, n_j\} \) such that \( a_k = 0 \) whenever \( m_j < k < n_j \) for some \( j \). If \( q \in \mathbb{T} \) is a regular point of \( f \), then the sequence \( \{\sum_{k=0}^{m_j} a_k z^k\} \) converges in some open neighborhood of \( q \) as \( j \to \infty \).

Thus, the sequence of partial sums \( \{s_m(z) := \sum_{k=0}^m a_k z^k\} \) has a subsequence \( \{s_{m_j}(z)\} \) which converges at some points \( z \) with \( |z| > 1 \), even though the full sequence \( \{s_m(z)\} \) certainly diverges whenever \( |z| > 1 \). The phenomenon is often referred to as over-convergence.

**Proof.** Replacing \( f(z) \) by \( f(qz) \), we may assume \( q = 1 \). Let \( \mathbb{D}(1, r) \) be a disk in which \( f \) extends holomorphically, and set \( U = \mathbb{D} \cup \mathbb{D}(1, r) \). Fix an integer \( N > 0 \) which satisfies \( \lambda > 1 + 1/N \), where \( \lambda \) is the constant given by Definition 7.7.

Consider the polynomial
\[
P(w) = \frac{1}{2} w^N(w + 1).
\]

Then \( P(1) = 1 \) and \( |P(w)| < 1 \) if \( |w| \leq 1 \) but \( w \neq 1 \). By continuity of \( P \), there exists an \( \varepsilon > 0 \) such that \( P(\mathbb{D}(0, 1 + \varepsilon)) \subset U \) (see Fig. 7). Define the holomorphic function \( g : \mathbb{D}(0, 1 + \varepsilon) \to \mathbb{C} \) by \( g = f \circ P \), and represent it by a power series
\[
(1) \quad g(w) = f(P(w)) = \sum_{k=0}^\infty b_k w^k \quad |w| < 1 + \varepsilon.
\]

On the other hand, the power series of \( f \) gives the representation
\[
(2) \quad g(w) = \sum_{k=0}^\infty a_k (P(w))^k \quad |w| < 1.
\]

so (1) and (2) must have identical coefficients. Note that the highest and lowest powers of \( w \) in \( (P(w))^k = 2^{-k} w^k N(w + 1)^k \) are \( k(n + 1) \) and \( kN \). We have \( n_j > \lambda m_j > (1 + 1/N)m_j \), or \( m_j(N + 1) < n_jN \), which shows

\[
\text{highest power of } w \text{ in } (P(w))^{m_j} < \text{lowest power of } w \text{ in } (P(w))^{n_j}.
\]

It follows from this, (1), (2), and the assumption \( a_k = 0 \) for \( m_j < k < n_j \) that
\[
\sum_{k=0}^{m_j(N+1)} b_k w^k = \sum_{k=0}^{m_j} a_k (P(w))^k \quad \text{for all } j.
\]

The left side converges for every \( w \in \mathbb{D}(0, 1 + \varepsilon) \) as \( j \to \infty \), so the same must be true of the right side. It follows that \( \sum_{k=0}^{m_j} a_k z^k \) converges for every \( z \in P(\mathbb{D}(0, 1 + \varepsilon)) \) as \( j \to \infty \). This completes the proof since \( P(\mathbb{D}(0, 1 + \varepsilon)) \) is an open set containing 1.

\[\square\]
The following classical result is the an immediate corollary:

**Theorem 7.9** (Hadamard’s Gap Theorem, 1892). Let \(\{m_j\}\) be a sequence of positive integers which satisfies \(m_{j+1} > \lambda m_j\) for some \(\lambda > 1\). If the power series \(f(z) = \sum_{k=0}^{\infty} a_k z^{m_k}\) has radius of convergence 1, then \(\mathbb{T}\) is the natural boundary of \(f\).

**Proof.** As before, let \(s_m\) denote the \(m\)-th partial sum of the power series of \(f\). The assumption shows that \((\{m_j\}, \{m_{j+1}\})\) is an Ostrowski pair and the subsequence \(\{s_m\}\) coincides with the full sequence \(\{s_m\}\) except for repetition of terms. If \(f\) had a regular point on \(\mathbb{T}\), it would follow from Theorem 7.8 that \(\{s_m\}\) converges in some neighborhood of that point, which is clearly impossible. \(\square\)

**Example 7.10.** Having \(\mathbb{T}\) as the natural boundary of \(f \in \mathcal{O}(\mathbb{D})\) is not necessarily an indication of discontinuity or non-differentiability of \(f\) on \(\mathbb{T}\); there could be a deeper analytic issue involved. Let \(f(z) = \sum_{k=0}^{\infty} \exp(-2^{k/2}) z^{2^k}\). Since

\[
\limsup_{k \to \infty} [\exp(-2^{k/2})]^{2^{-k}} = \limsup_{k \to \infty} \exp(-2^{-k/2}) = 1,
\]
the series has radius of convergence 1. Hadamard’s Gap Theorem 7.9, with \(m_j = 2^j\), shows that \(\mathbb{T}\) is the natural boundary of \(f\). Term-by-term differentiation gives

\[
f^{(n)}(z) = \sum_{2^k \geq n} 2^k (2^k - 1) \cdots (2^k - n + 1) \exp(-2^{k/2}) z^{2^k - n} \quad n \geq 1.
\]

The series converges uniformly on \(\mathbb{D}\) since

\[
|2^k (2^k - 1) \cdots (2^k - n + 1) \exp(-2^{k/2}) z^{2^k - n}| \leq n 2^k \exp(-2^{k/2}) \quad \text{if } |z| \leq 1
\]

and \(\sum_{2^k \geq n} 2^k \exp(-2^{k/2})\) converges. It follows that \(f\) and all of its derivatives extend continuously to \(\mathbb{D}\). In particular, \(f|_{\mathbb{T}}\) is \(C^\infty\) smooth.

### §7.2. Analytic continuation along curves

**Definition 7.11.** By a **function element** is meant a pair \((f, D)\), where \(D\) is an open disk in \(\mathbb{C}\) and \(f \in \mathcal{O}(D)\). We say that the function element \((f_1, D_1)\) is **(direct) analytic continuation** of another function element \((f_0, D_0)\) if \(D_0 \cap D_1 \neq \emptyset\) and \(f_0 = f_1\) in \(D_0 \cap D_1\). In this case, we write \((f_0, D_0) \sim (f_1, D_1)\).

The essential property of disks that we use here is that either \(D_0 \cap D_1\) is empty or else connected. The relation \(\sim\) is clearly reflexive and symmetric but not transitive, as the following example shows.
Example 7.12. Let $\omega = e^{2\pi i/3}$ and consider the disks $D_k := \mathbb{D}(\omega^k, 1)$ for $k = 0, 1, 2$ (see Fig. 1). Consider the holomorphic branches $f_k \in \mathcal{O}(D_k)$ of $\sqrt[3]{z}$ defined in polar coordinates by
\[
\begin{align*}
    f_0(re^{it}) &= \sqrt[3]{r}e^{it/2} & \frac{\pi}{2} < t < \frac{\pi}{2}, \\
    f_1(re^{it}) &= \sqrt[3]{r}e^{it/2} & \frac{\pi}{6} < t < \frac{7\pi}{6}, \\
    f_2(re^{it}) &= \sqrt[3]{r}e^{it/2} & \frac{5\pi}{6} < t < \frac{11\pi}{6}.
\end{align*}
\]

The $(f_0, D_0) \sim (f_1, D_1)$ and $(f_1, D_1) \sim (f_2, D_2)$ but $(f_0, D_0) \not\sim (f_2, D_2)$ since in $D_0 \cap D_2$ we have $f_0 = -f_2$.

However, the following form of transitivity holds:

Lemma 7.13. If the triple intersection $D_0 \cap D_1 \cap D_2$ is non-empty, then the relations $(f_0, D_0) \sim (f_1, D_1)$ and $(f_1, D_1) \sim (f_2, D_2)$ imply $(f_0, D_0) \sim (f_2, D_2)$.

Proof. By the Identity Theorem, $f_0 = f_1 = f_2$ in $D_0 \cap D_1 \cap D_2$ implies $f_0 = f_2$ in $D_0 \cap D_2$. \qed
Definition 7.14. A chain of function elements is a collection \((f_k, D_k)_{0 \leq k \leq n}\) such that \((f_k, D_k) \sim (f_{k+1}, D_{k+1})\) for all \(0 \leq k \leq n - 1\). Given two function elements \((f_0, D_0)\) and \((f_n, D_n)\), let \(\gamma : [0, 1] \to \mathbb{C}\) be a curve such that \(\gamma(0)\) is the center of \(D_0\) and \(\gamma(1)\) is the center of \(D_n\). We say that \((f_n, D_n)\) is obtained from \((f_0, D_0)\) by analytic continuation along \(\gamma\) if there is a chain \((f_k, D_k)_{0 \leq k \leq n}\) and points \(0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1\) such that \(\gamma([t_k, t_{k+1}]) \subset D_k\) for all \(0 \leq k \leq n\).

Lemma 7.15. Consider a function element \((f_0, D_0)\) and a curve \(\gamma : [0, 1] \to \mathbb{C}\) whose initial point \(\gamma(0)\) is the center of \(D_0\). If the function elements \((g_n, A_n)\) and \((h_m, B_m)\) are both obtained from \((f_0, D_0)\) by analytic continuation along \(\gamma\), then \((g_n, A_n) \sim (h_m, B_m)\).

Proof. Let \((g_j, A_j)_{0 \leq j \leq n}\) and \((h_k, B_k)_{0 \leq k \leq m}\) be two chains with \((g_0, A_0) = (h_0, B_0) = (f_0, D_0)\), and \(0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1\) and \(0 = s_0 < s_1 < \cdots < s_n < s_{m+1} = 1\) be points such that

\[
\gamma([t_j, t_{j+1}]) \subset A_j \quad \text{if} \quad 0 \leq j \leq n
\]

\[
\gamma([s_k, s_{k+1}]) \subset B_k \quad \text{if} \quad 0 \leq k \leq m.
\]

We claim that if \([t_j, t_{j+1}] \cap [s_k, s_{k+1}] \neq \emptyset\), then \((g_j, A_j) \sim (h_k, B_k)\). Suppose this is not the case, and choose such pairs with \(j + k\) minimal. Evidently, \(j + k > 0\). Without loss of generality, we can assume \(t_j \leq s_k \leq t_{j+1}\). Then \(\gamma(s_k) \in B_{k-1} \cap B_k \cap A_j\). By minimality, \((g_j, A_j) \sim (h_{k-1}, B_{k-1})\). Since \((h_{k-1}, B_{k-1}) \sim (h_k, B_k)\) and since \(B_{k-1} \cap B_k \cap A_j \neq \emptyset\), Lemma 7.13 shows that \((g_j, A_j) \sim (h_k, B_k)\), which is a contradiction. This proves the claim. In particular, since \([t_n, t_{n+1}] \cap [s_m, s_{m+1}] \neq \emptyset\), we must have \((g_n, A_n) \sim (h_m, B_m)\).

Theorem 7.16 (Homotopy invariance of analytic continuation). Let \(\{\gamma_s\}_{0 \leq s \leq 1}\) be a homotopy between the curves \(\gamma_0, \gamma_1 : [0, 1] \to \mathbb{C}\). Suppose a function element \((f, D)\) can be continued analytically along every \(\gamma_s\) to some function element \((f_s, D_s)\). Then \((f_0, D_0) \sim (f_1, D_1)\).

Proof. Fix \(0 \leq s \leq 1\), a chain \((g_k, A_k)_{0 \leq k \leq n}\) with \((g_0, A_0) = (f, D)\) and \((g_n, A_n) = (f_s, D_s)\), and points \(0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1\) such that \(\gamma_s([t_k, t_{k+1}]) \subset A_k\) for all \(0 \leq k \leq n\). Uniform continuity of the homotopy \((t, s) \mapsto \gamma_s(t)\) on \([0, 1] \times [0, 1]\) shows that there exists an open neighborhood \(I_s\) of \(s\) with the property that the same chain works for \(\gamma_r\) if \(r \in I_s \cap [0, 1]\), in the sense that \(\gamma_r([t_k, t_{k+1}]) \subset A_k\) for all \(0 \leq k \leq n\). It follows from Lemma 7.13 that \((f_s, D_s) \sim (f_r, D_r)\) if \(r \in I_s \cap [0, 1]\). Since \([0, 1]\) can be covered by a finite number of neighborhoods of the form \(I_s\), it follows that \((f_0, D_0) \sim (f_1, D_1)\).
Theorem 7.17 (Monodromy Theorem). Let \( U \subset \mathbb{C} \) be a simply-connected domain containing a disk \( D \). Suppose a function element \( (f, D) \) can be analytically continued along every curve in \( U \) whose initial point is the center of \( D \). Then there exists \( F \in \mathcal{O}(U) \) such that \( F = f \) in \( D \).

**Proof.** Let \( z \in U \) and take any curve \( \gamma : [0, 1] \to U \) with \( \gamma(0) \) equal to the center of \( D \) and \( \gamma(1) = z \). Let \((g, A)\) be a function element obtained by analytic continuation of \((f, D)\) along \( \gamma \). Define \( F(z) := g(z) \). Since \( U \) is simply-connected, any curve in \( U \) which joins the center of \( D \) to \( z \) is homotopic to \( \gamma \), so by Theorem 7.16 \( F(z) \) does not depend on the choice of \( \gamma \). The construction shows that in fact \( F = g \) in \( A \). It easily follows that \( F \) is holomorphic in \( U \) and \( F = f \) in \( D \). \( \square \)

**Problems**

1. Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function which sends real numbers to real numbers and imaginary numbers to imaginary numbers. Prove that \( f \) is an odd function:
   \[ f(-z) = -f(z) \quad \text{for all } z \in \mathbb{C}. \]

2. Let \( f \) be a holomorphic function defined in an open neighborhood of the origin which satisfies
   \[ f(2z) = (f(z))^2 \]
   for all \( z \) sufficiently close to 0. Use this functional equation to show that \( f \) can be extended to an entire function. Can you determine all such entire functions explicitly? (Hint: Study the cases \( f(0) = 0 \) and \( f(0) = 1 \) separately.)

3. Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) have radius of convergence 1, and define
   \[ g(w) = \frac{1}{1 - w} f \left( \frac{w}{1 - w} \right). \]
   (i) Show that \( g \) is holomorphic in \( \{w : \text{Re}(w) < 1/2\} \).
   (ii) If \( g(w) = \sum_{k=0}^{\infty} b_k w^k \), show that
   \[ b_k = \sum_{n=0}^{k} \binom{k}{n} a_n, \]
   where, as usual, \( \binom{k}{n} = k! / (n!(k-n)!) \).
   (iii) Show that \( z = 1 \) is a singular point for \( f \) if and only if \( w = 1/2 \) is a singular point for \( g \) if and only if \( \limsup_{k \to \infty} \sqrt[k]{|b_k|} = 2 \).

4. (Pringsheim) Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) have radius of convergence \( R = 1 \), and \( a_k \geq 0 \) for all \( k \geq 0 \). Show that \( z = 1 \) is a singular point of \( f \). (Hint: Otherwise \( f \) could be analytically continued in a small disk around \( z = 1 \). Then the power series of \( f \) about \( \frac{1}{2} \) would converge in a disk \( D(1/2, 1/2 + \epsilon) \) for a small \( \epsilon > 0 \). Since \( a_k \geq 0 \), for every \( 0 \leq t \leq 2\pi \), the power series of \( f \) about \( e^{it}/2 \) would converge in \( D(e^{it}/2, 1/2 + \epsilon) \), which would contradict \( R = 1 \).)
(5) Let $P = P(z, w)$ be a polynomial with complex coefficients in two variables. Suppose that $(f_0, D)$ and $(g_0, D)$ are two function elements such that $P(f_0, g_0) = 0$ everywhere in $D$. If $(f_1, A)$ and $(g_1, A)$ are analytic continuations of $(f_0, D)$ and $(g_0, D)$ along the same curve, show that $P(f_1, g_1) = 0$ everywhere in $A$. Thus, the relation $P(\cdot, \cdot) = 0$ persists under analytic continuation.

(6) Use the Monodromy Theorem 7.17 to give another proof for Theorem 6.8. If $U \subset \mathbb{C}$ is a simply connected domain and $\varphi : U \to \mathbb{R}$ is harmonic, then there exists an $f \in \mathcal{O}(U)$ such that $\text{Re}(f) = \varphi$. 