Math 748 Problems

(1) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that $|f'(0)| \leq 1$ (this is Schwarz lemma if $f(0) = 0$). Conclude that if $f : \mathbb{D}(p, \varepsilon) \to \mathbb{D}(q, \delta)$ is holomorphic, then $|f'(p)| \leq \delta/\varepsilon$.

(2) Let f and g be non-identity elements in $\mathrm{PSL}_2(\mathbb{C})$. Show that f is conjugate to g iff $\mathrm{tr}^2(f) = \mathrm{tr}^2(g)$. Verify that a non-identity $f \in \mathrm{PSL}_2(\mathbb{C})$ is

- hyperbolic iff $\mathrm{tr}^2(f) > 4$;
- parabolic iff $\mathrm{tr}^2(f) = 4$;
- elliptic iff $0 \leq \mathrm{tr}^2(f) < 4$;
- loxodromic iff $\mathrm{tr}^2(f) \in \mathbb{C} \smallsetminus [0, +\infty[$.

(3) Show that an element of $\mathrm{PSL}_2(\mathbb{C})$ corresponds to a rigid rotation of the sphere (under the stereographic projection) iff it has the form

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}.$$

Note that this gives another proof for $\mathrm{SO}(3) \cong \mathbb{RP}^3$ since the space of all such matrices is homeomorphic to the 3-sphere $\{(a, b) \in \mathbb{C}^2 : |a|^2 + |b|^2 = 1\}$ with the antipodal points (a, b) and $(-a, -b)$ identified.

(4) (*Iwasawa decomposition*) Show that every $f \in \mathrm{PSL}_2(\mathbb{C})$ can be written in a unique way as

$$f = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},$$

where $a, b, \mu \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, and $\lambda > 0$. Conclude that $\mathrm{PSL}_2(\mathbb{C})$ is homeomorphic to $\mathbb{RP}^3 \times \mathbb{R}^3$.

(5) Show that the upper half-plane $\mathbb{H}$ can be identified with the subgroup of $\mathrm{Aut}(\mathbb{H})$ consisting of real affine transformations. Use this identification to make $\mathbb{H}$ into a Lie group. Show that under this identification, every left multiplication $L_z : w \mapsto z * w$ acts as an automorphism of $\mathbb{H}$. What are the left and right invariant volume forms of the Lie group $(\mathbb{H}, *)$?

(6) Show that if $f \in \mathrm{Aut}(\mathbb{H})$ and $z, w \in \mathbb{H}$, then

$$|f(z) - f(w)| = |z - w| \, |f'(z) f'(w)|^{\frac{1}{2}}.$$

(7) Let Γ be a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Prove that the following conditions are equivalent:

(i) Γ acts properly discontinuously on $\mathbb{H}$;

(ii) For every $x \in \mathbb{H}$, the orbit $\Gamma x = \{\gamma(x) : \gamma \in \Gamma\}$ is a discrete subset of $\mathbb{H}$;

(iii) Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

(8) Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ all of whose non-identity elements have the same fixed-point set. Prove that Γ is cyclic.

(9) Show that no discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ can be isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

(10) Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Show that the normalizer of Γ in $\mathrm{PSL}_2(\mathbb{R})$ defined by

$$N(\Gamma) = \{\gamma \in \mathrm{PSL}_2(\mathbb{R}) : \gamma\Gamma\gamma^{-1} \subset \Gamma\}$$

is discrete iff Γ is non-Abelian.

(11) Let X, Y be Riemann surfaces, $\pi_X : \widetilde{X} \to X$ and $\pi_Y : \widetilde{Y} \to Y$ be their universal coverings, and Γ_X and Γ_Y be the corresponding deck groups. Consider a holomorphic map $f : X \to Y$.

(i) Show that f lifts to a holomorphic map $F : \widetilde{X} \to \widetilde{Y}$ so that $f \circ \pi_X = \pi_Y \circ F$. Verify that F is unique up to postcomposition with an element of Γ_Y.

(ii) Show that f induces a group homomorphism $\theta : \Gamma_X \to \Gamma_Y$ which satisfies $\theta(\gamma) \circ F = F \circ \gamma$ for every $\gamma \in \Gamma_X$.

(iii) Assume now that f is a biholomorphism. Prove that F is a biholomorphism and that the conjugation $\theta : \gamma \mapsto F \circ \gamma \circ F^{-1}$ is a group isomorphism $\Gamma_X \xrightarrow{\cong} \Gamma_Y$.

(12) Let $\widetilde{X}$ be a simply-connected Riemann surface and $\Gamma_1, \Gamma_2 \subset \mathrm{Aut}(\widetilde{X})$ be fixed-point free properly discontinuous subgroups. Show that the Riemann surfaces $\widetilde{X}/\Gamma_1$ and $\widetilde{X}/\Gamma_2$ are biholomorphic iff there exists $F \in \mathrm{Aut}(\widetilde{X})$ such that $\Gamma_2 = F\,\Gamma_1\,F^{-1}$.

(13) Let Γ be a fixed-point free discrete subgroup of $\mathrm{Aut}(\mathbb{D})$. Show that the automorphism group of the Riemann surface $\mathbb{D}/\Gamma$ is isomorphic to the quotient $N(\Gamma)/\Gamma$.

(14) For $\tau \in \mathbb{H}$, denote by Γ_τ the subgroup of $\mathrm{Aut}(\mathbb{C})$ generated by $z \mapsto z+1$ and $z \mapsto z+\tau$. Show that any torus $\mathbb{C}/\Gamma$ is biholomorphic to some $\mathbb{C}/\Gamma_\tau$, but this τ is far from being unique. In fact, show that the tori $\mathbb{C}/\Gamma_\tau$ and $\mathbb{C}/\Gamma_\sigma$ are biholomorphic iff

$$\sigma = \frac{a\tau + b}{c\tau + b}, \qquad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

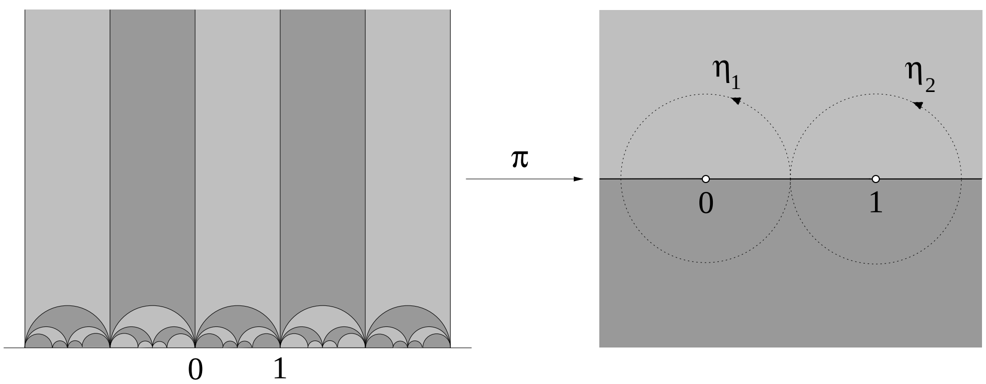
It follows that the "moduli space" of compact Riemann surfaces of genus 1 is isomorphic to the quotient $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$.

(15) Let $\pi : \mathbb{H} \to \mathbb{C} \smallsetminus \{0,1\}$ be the universal covering map given by the standard "elliptic modular function" (recall that π is obtained by mapping the ideal triangle with vertices at $0, 1, \infty$ conformally onto $\mathbb{H}$ and extending it by reflections through the edges of this triangle in the domain and the real line in the range).

(i) Consider the generators of the fundamental group of $\mathbb{C} \smallsetminus \{0,1\}$ represented by

$$\eta_1(t) = \frac{1}{2}e^{it}, \quad \eta_2(t) = -\frac{1}{2}e^{it} + 1 \quad (0 \leq t \leq 2\pi),$$

as shown in the figure. Find the explicit formulas of the deck transformations in $\mathrm{Aut}(\mathbb{H})$ corresponding to these generators.



(ii) Show that the deck group $\Gamma \subset \mathrm{Aut}(\mathbb{H})$ associated with π is isomorphic to the "modular group"

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (\mathrm{mod} 2) \right\}.$$

(16) This problem gives two practically useful methods of proving that a given map is covering. Let X and Y be connected topological manifolds.

(i) Recall that $f : Y \to X$ is *proper* if $f^{-1}(K)$ is compact for every compact set $K \subset X$. Show that $f : Y \to X$ is a finite-degree covering map iff it is a proper local homeomorphism.

(ii) Let us say that a continuous map $f : Y \to X$ has the *curve lifting property* if given any curve $\gamma : [0,1] \to X$ and any $y \in Y$ with $f(y) = \gamma(0)$ there exists a lift $\widetilde{\gamma} : [0,1] \to Y$ of γ such that $\widetilde{\gamma}(0) = y$. Show that $f : Y \to X$ is a covering map iff it is a local homeomorphism with the curve lifting property.

(17) Let $f : X \to Y$ be a holomorphic map between hyperbolic Riemann surfaces and $F : \mathbb{D} \to \mathbb{D}$ be any lift of f to the universal coverings. Show that $F \in \mathrm{Aut}(\mathbb{D})$ iff f is a covering map.

(18) The maps $E : \mathbb{H} \to \mathbb{D}^*$ and $\Pi_k : \mathbb{D}^* \to \mathbb{D}^*$ given by

$$E(z) = \exp(2\pi i z) \qquad \text{and} \qquad \Pi_k(z) = z^k$$

evidently define coverings of the punctured disk. This problem shows that up to biholomorphism, these are the only holomorphic coverings of $\mathbb{D}^*$. Let X be a Riemann surface and $\pi : X \to \mathbb{D}^*$ be a holomorphic covering map.

(i) If π has infinite degree, show that there exists a biholomorphic map $f : X \to \mathbb{H}$ such that $\pi = E \circ f$.

(ii) If π has finite degree k, show that there exists a biholomorphic map $f : X \to \mathbb{D}^*$ such that $\pi = \Pi_k \circ f$.

(19) Prove that every hyperbolic Riemann surface with Abelian fundamental group is biholomorphic to the unit disk $\mathbb{D}$, or to the punctured disk $\mathbb{D}^*$, or to an annulus $\mathbb{A}(1, R) = \{z : 1 < |z| < R\}$ for a unique $R > 1$.

(20) Show that given any two unit (in hyperbolic norm) tangent vectors $\mathbf{u}, \mathbf{v} \in T\mathbb{H}$ there exists a unique $f \in \mathrm{Aut}(\mathbb{H})$ such that $f_*(\mathbf{u}) = \mathbf{v}$. Conclude that the unit tangent bundle of $\mathbb{H}$ can be identified with $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$.

(21) Show that every non-identity $f \in \mathrm{PSL}_2(\mathbb{R})$ belongs to a unique one-parameter subgroup (=homomorphic image of the additive group of real numbers) and that this one-parameter subgroup is conjugate to

$$t \mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

depending on whether f is hyperbolic, parabolic, or elliptic. If you identify $\mathrm{PSL}_2(\mathbb{R})$ with the unit tangent bundle of $\mathbb{H}$, can you visualize a one-parameter subgroup geometrically?

(22) This problem discusses invariant metrics on non-hyperbolic Riemann surfaces.

(i) Show that there is no Riemannian metric on $\widehat{\mathbb{C}}$ which is invariant under the action of $\mathrm{Aut}(\widehat{\mathbb{C}})$. However, verify that the *spherical metric*

$$\sigma = \frac{2\,|dz|}{1 + |z|^2}$$

on $\widehat{\mathbb{C}}$ is invariant under the action of the subgroup $\mathrm{SO}(3) \subset \mathrm{Aut}(\widehat{\mathbb{C}})$ consisting of all rotations $z \mapsto \dfrac{az + b}{-\overline{b}z + \overline{a}}$.

(ii) Similarly, show that there is no Riemannian metric on $\mathbb{C}$ which is invariant under the action of $\mathrm{Aut}(\mathbb{C})$. However, prove that every parabolic Riemann surface admits a flat Euclidean metric which is unique up to multiplication by a positive constant.

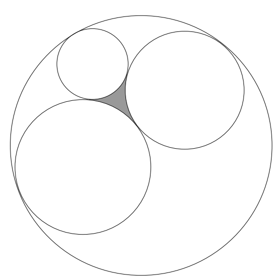
(23) Show that $f \in \mathrm{Aut}(\mathbb{H})$ is hyperbolic iff it carries a (necessarily unique) Poincaré geodesic to itself without fixed points. This geodesic is often called the *axis* of f. Verify that f acts on its axis as a translation. What is the relation between the hyperbolic length of this translation and the invariant $\mathrm{tr}^2(f)$?

(24) Verify that the hyperbolic distances in the unit disk and the upper half-plane are given by

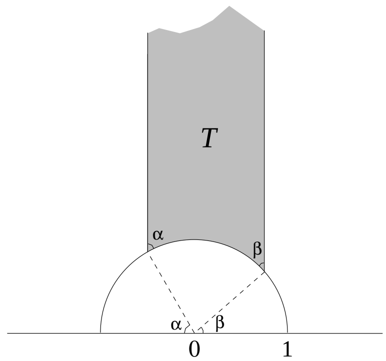
$$\mathrm{dist}_{\mathbb{D}}(z,w) = \log\left(\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}\right),$$
$$\mathrm{dist}_{\mathbb{H}}(z,w) = \log\left(\frac{|z-\overline{w}|+|z-w|}{|z-\overline{w}|-|z-w|}\right).$$

(25) Show that a given pair of points in $\mathbb{D}$ can be mapped by an element of $\mathrm{Aut}(\mathbb{D})$ to another pair of points iff they have the same hyperbolic distance. By contrast, show that three given points on the boundary circle $\partial\mathbb{D}$ can be mapped by an element of $\mathrm{Aut}(\mathbb{D})$ to another three points on $\partial\mathbb{D}$ iff they have the same cyclic order.

(26) Three circles in $\overline{\mathbb{D}}$ are tangent to each other and to the unit circle $\partial\mathbb{D}$ as shown. Prove that the hyperbolic area of the shaded region does not depend on the particular choice of the circles.



(27) (*Gauss-Bonnet in hyperbolic plane*) Let T be a hyperbolic triangle with vertices in $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ and with interior angles α, β, γ. Show that the hyperbolic area of T is $\pi - (\alpha+\beta+\gamma)$. You may find it easier to first consider the case where $\gamma = 0$ so that the corresponding vertex is on $\mathbb{R} \cup \{\infty\}$. In this case, after applying an automorphism (which does not change the area or angles), you can put T is the position shown and compute the area by integration.



(28) For an ordered quadruple (z_1, z_2, z_3, z_4) of distinct points in $\widehat{\mathbb{C}}$, define the *cross-ratio* by

$$\chi(z_1, z_2, z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_2 - z_1)(z_4 - z_3)} \in \mathbb{C} \setminus \{0, 1\}.$$

(i) Let $f \in \mathrm{Aut}(\widehat{\mathbb{C}})$ be the unique automorphism which sends z_1, z_2, z_4 to $0, 1, \infty$, respectively. Show that $f(z_3) = \chi(z_1, z_2, z_3, z_4)$.

(ii) Verify that if $f \in \mathrm{Aut}(\widehat{\mathbb{C}})$, then

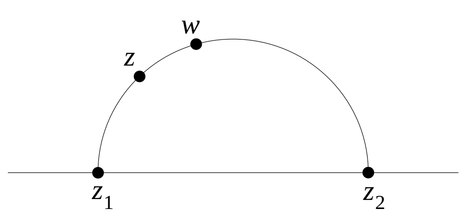
$$\chi(f(z_1), f(z_2), f(z_3), f(z_4)) = \chi(z_1, z_2, z_3, z_4)$$

for every quadruple (z_1, z_2, z_3, z_4). Conversely, show that if a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ preserves cross-ratios of all quadruples, then $f \in \mathrm{Aut}(\widehat{\mathbb{C}})$.

(iii) Show that $\chi(z_1, z_2, z_3, z_4) \in \mathbb{R}$ iff the points z_1, z_2, z_3, z_4 lie on a circle.

(iv) Let z, w be points in $\mathbb{H}$ and suppose that the geodesic joining z to w meets $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ at z_1 and z_2, with z between z_1 and w as shown. Prove that

$$\mathrm{dist}_{\mathbb{H}}(z, w) = \log \chi(z_1, z, w, z_2).$$



(29) Let $L \subset \mathbb{H}$ be a hyperbolic geodesic. Describe the set of points in $\mathbb{H}$ whose hyperbolic distance to L is at most $d > 0$.

(30) Compute the hyperbolic perimeter $L(r)$ and area $A(r)$ of a hyperbolic circle of radius $r > 0$. What is $\lim_{r \to +\infty} L(r)/A(r)$?

(31) Two particles are at the point 0 of the hyperbolic plane $\mathbb{D}$. At time $t = 0$ they start moving with unit speed along two geodesics through 0 which make an angle θ. Let $\ell(t)$ be their distance at time $t > 0$. Show that

$$2t - \ell(t) \to c(\theta) \qquad \text{as } t \to +\infty,$$

where $c(\theta)$ depends only on θ. Compare this with the corresponding situation in the Euclidean plane, where $2t - \ell(t)$ increases linearly in t (unless $\theta = \pi$).

(32) Show that triangles in the hyperbolic plane are peculiarly "thin" in the following sense: If ABC is any hyperbolic triangle in $\mathbb{D}$ and $z \in BC$, then there exists a $w \in AB \cup AC$ such that

$$\mathrm{dist}_{\mathbb{D}}(z, w) \leq \log(1 + \sqrt{2}) \approx 0.881373.$$

This phenomenon is common in spaces of negative curvature.

(33) Show that the hyperbolic metric on $\mathbb{D}^*$ is given by

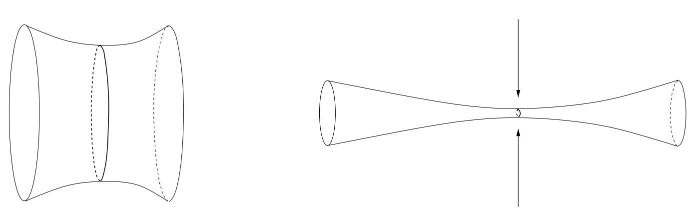
$$\rho_{\mathbb{D}^*} = \frac{-1}{|z| \log |z|}\, |dz|,$$

while on the annulus $\mathbb{A}(1, R) = \{z : 1 < |z| < R\}$ it takes the form

$$\rho_{\mathbb{A}(1,R)} = \frac{\frac{\pi}{\log R}}{|z| \sin\left(\frac{\pi \log |z|}{\log R}\right)}\, |dz|.$$

(34) Show that there is no closed hyperbolic geodesic in $\mathbb{D}^*$. On the other hand, show that for the annulus $\mathbb{A}(1, R)$ the circle $|z| = \sqrt{R}$ is a closed geodesic, and that it is the unique closed geodesic in its free homotopy class.

(35) Let X be a *conformal annulus*, that is, a Riemann surface biholomorphic to the round annulus $\mathbb{A}(1, R)$ for some (necessarily unique!) $R > 1$. Recall that the *modulus* of X is defined as $\mathrm{mod}(X) = \frac{1}{2\pi} \log R$. Let η be the *core curve* of X, that is, the unique closed geodesic which generates $\pi_1(X)$. Show that the hyperbolic length of η is given by $\pi/\mathrm{mod}(X)$. In particular, squeezing the core curve of X makes X look thicker.



(36) Let X be a hyperbolic Riemann surface.

(i) Show that if η is a curve in X connecting p to q, then there exists a unique geodesic connecting p to q which is homotopic to η rel $\{p,q\}$. In particular, if η is a loop passing through p, there exists a unique geodesic loop passing through p homotopic to η rel $\{p\}$ (note that this geodesic loop may not be a closed geodesic; it may have a corner at p).

(ii) Show by an example that there may be no closed geodesic homotopic a given loop on X. (However, it can be shown that when X is *compact*, there is a unique closed geodesic homotopic to a given loop η, and this geodesic is simple if η is.)

(37) Suppose X is a hyperbolic Riemann surface with the universal covering map $\pi : \mathbb{D} \to X$. For $z, w \in X$, show that

$$\mathrm{dist}_X(z,w) = \inf \{\mathrm{dist}_{\mathbb{D}}(\widetilde{z}, \widetilde{w}) : \pi(\widetilde{z}) = z \text{ and } \pi(\widetilde{w}) = w\}.$$

(38) Suppose X is a hyperbolic Riemann surface with the universal covering map $\pi : \mathbb{D} \to X$. For $z \in X$ define

$$r(z) = \frac{1}{2} \inf \{\mathrm{dist}_{\mathbb{D}}(\widetilde{z}, \widetilde{w}) : \pi(\widetilde{z}) = z = \pi(\widetilde{w}) \text{ and } \widetilde{z} \neq \widetilde{w}\}.$$

(i) For $z \in X$ and $\widetilde{z} \in \pi^{-1}(z)$, show that the π maps the hyperbolic ball $B_{\mathbb{D}}(\widetilde{z}, r(z))$ homeomorphically onto the hyperbolic ball $B_X(z, r(z))$. In particular, $B_X(z, r(z))$ is an embedded disk. Because of this property, $r(z)$ is often called the *injectivity radius* at z.

(ii) Show that $r(z)$ is the largest radius for which (i) holds. In fact, verify that the closure of $B_X(z, r(z))$ contains a homotopically non-trivial loop, so $B_X(z, \delta)$ in not an embedded disk if $\delta > r(z)$. Conclude that $2r(z)$ is the length of the shortest homotopically non-trivial loop in X passing through z.

(iii) Verify that $r : X \to \mathbb{R}^+$ is 1-Lipschitz, that is,

$$|r(z) - r(w)| \leq \mathrm{dist}_X(z,w) \qquad \text{for all } z, w \in X.$$

(39) Let X be a compact hyperbolic Riemann surface (so that X has genus ≥ 2).

(i) Show that there exists $\varepsilon > 0$ such that every loop on X of hyperbolic length $< \varepsilon$ is homotopically trivial.

(ii) Let $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ be the deck group of the universal covering map $\pi : \mathbb{D} \to X$. Show that every element of Γ is a hyperbolic automorphism.

(40) (*Horoballs*) For any $p, z \in \mathbb{D}$, let $B(z)$ be the hyperbolic ball centered at z of radius $\mathrm{dist}_{\mathbb{D}}(z,p)$. Show that as $z \to w \in \partial\mathbb{D}$ while p is fixed, the ball $B(z)$ converges to a Euclidean ball H which is tangent to $\partial\mathbb{D}$ at w and has p on its

boundary. Such an H is called a *horoball* in the hyperbolic plane. Prove the following statements:

(i) An automorphism $\gamma : \mathbb{D} \to \mathbb{D}$ maps horoballs touching $\partial\mathbb{D}$ at w to horoballs touching $\partial\mathbb{D}$ at $\gamma(w)$.

(ii) Let H be a horoball touching $\partial\mathbb{D}$ at w. Then a non-identity $\gamma \in \mathrm{Aut}(\mathbb{D})$ satisfies $\gamma(H) = H$ iff γ is parabolic with fixed point w.

(iii) Let $\gamma \in \mathrm{Aut}(\mathbb{D})$ be parabolic with fixed point $w \in \partial\mathbb{D}$. Show that the centralizer

$$\mathrm{Cent}(\gamma) = \{\sigma \in \mathrm{Aut}(\mathbb{D}) : \gamma \circ \sigma = \sigma \circ \gamma\}$$

is precisely the subgroup of all automorphisms which preserve every horoball touching $\partial\mathbb{D}$ at w.

(iv) Let H be a horoball and $\gamma \neq \mathrm{id}$ be a parabolic automorphism with $\gamma(H) = H$. Show that there is an $0 < r < 1$ such that the quotient Riemann surface $H/\langle\gamma\rangle$ is isometric to the punctured disk $\mathbb{D}^*(0, r)$ equipped with the hyperbolic metric of $\mathbb{D}^*$. Conclude that this quotient has finite hyperbolic area. What is the relation between the area of $H/\langle\gamma\rangle$ and the length of the loop $\partial H/\langle\gamma\rangle$?

(41) Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ containing the parabolic element $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(i) Show that for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, either $c = 0$ or $|c| \geq 1$. To this end, you may find it useful to consider the sequence $\gamma_n \in \Gamma$ defined by

$$\gamma_0 = \gamma, \quad \gamma_{n+1} = \gamma_n \circ \tau \circ \gamma_n^{-1}.$$

(ii) Suppose Γ is a discrete subgroup of $\mathrm{Aut}(\mathbb{H})$ which contains the translation $\tau : z \mapsto z + b$ $(b > 0)$, and denote by $\mathrm{Cent}(\tau)$ the centralizer of τ in Γ, consisting of all translations in Γ. Show that the horoball $H = \{z : \Im(z) > b\}$ satisfies

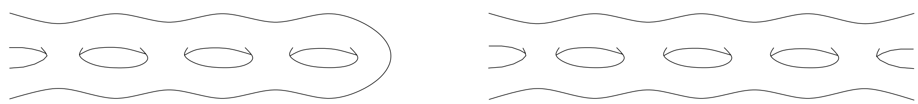
$$\sigma(H) \cap H = \emptyset \qquad \text{for all } \sigma \in \Gamma \smallsetminus \mathrm{Cent}(\tau).$$

(42) (*Ends*) Let X be a connected oriented topological surface and $\mathcal{K} = \{K_n\}$ be an exhaustion of X, that is, each $K_n \subset X$ is compact, $K_n \subset \mathrm{int}(K_{n+1})$ and $X = \bigcup K_n$. An *end sequence* $\{U_n\}$ associated with $\mathcal{K}$ is a choice of a non-empty connected component U_n of $X \smallsetminus K_n$ for each n, subject to the condition $U_{n+1} \subset U_n$. Two end sequences $\{U_n\}$ and $\{U'_m\}$ associated with exhaustions $\mathcal{K}$ and $\mathcal{K}'$ are considered equivalent if $U_n \cap U'_m \neq \emptyset$ for all n, m. Each equivalence class of this relation is called an *end* of X.

(i) Verify that X is compact iff it has no ends.

(ii) If X is embedded in a compact surface Y, show that there is a one-to-one correspondence between the ends of X and the connected components of $\partial X \subset Y$.

(iii) Show that the surface on the left has a unique end, while the one on the right has two ends. Conclude that these surfaces are not homeomorphic (try proving this without using the concept of end!).



(43) (*Cusps*) Suppose X is a hyperbolic Riemann surface and Γ is the deck group of the universal covering map $\mathbb{D} \to X$. An end of X is called a *cusp* if it is represented by an end sequence $\{U_n\}$, where each U_n is biholomorphic to $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$.

(i) If $\{U_n\}$ represents a cusp, show that $\pi_1(U_{n+1}) \cong \pi_1(U_n)$ and that the inclusion $U_n \hookrightarrow X$ injects $\pi_1(U_n)$ into $\pi_1(X)$.

(ii) If $\gamma \in \Gamma$ is the deck transformation corresponding to a generator of $\pi_1(U_n) \cong \mathbb{Z}$, show that γ must be a parabolic automorphism.

(iii) Conversely, show that every parabolic $\gamma \in \Gamma$ gives rise to a cusp in X by completing the following sketch: The centralizer of γ in Γ is infinite cyclic with a parabolic generator γ_1. Let $H = \gamma_1(H)$ be a horoball such that $\sigma(H) \cap H = \emptyset$ for all $\sigma \in \Gamma \smallsetminus \langle \gamma_1 \rangle$. Choose a sequence $H \supset H_1 \supset H_2 \supset \cdots$ of horoballs all touching $\partial\mathbb{D}$ at the same point as H, and show that $\{U_n = H_n/\Gamma\}$ represents a cusp in X.

(iii) Conclude that every cusp is represented by an end sequence $\{U_n\}$, where each (U_n, ρ_X) is isometric to $(\mathbb{D}^*(0, r_n), \rho_{\mathbb{D}^*})$ for some $0 < r_n < 1$. In other words, a neighborhood of 0 in the punctured disk is a universal model for a neighborhood of a cusp on a hyperbolic surface.

It can be shown that any end represented by a sequence $\{U_n\}$ in which some U_n has finite hyperbolic area is in fact a cusp.

(44) Recall that the *Gaussian curvature* of a C^2 conformal metric $\rho = \rho(z)\,|dz|$ on a Riemann surface is defined by

$$\kappa(z) = -\frac{\Delta \log \rho(z)}{\rho^2(z)},$$

where $\Delta = 4\dfrac{\partial}{\partial z}\dfrac{\partial}{\partial \overline{z}}$ is the usual Laplacian.

(i) Verify that this definition makes κ conformally invariant. In other words, if $z \mapsto w$ is conformal and $\widetilde{\rho}(w)|dw|$ is the corresponding metric in the coordinate w, then

$$-\frac{\Delta \log \rho(z)}{\rho^2(z)} = -\frac{\Delta \log \widetilde{\rho}(w)}{\widetilde{\rho}^{\,2}(w)}.$$

In particular, the definition of κ does not depend on the choice of the coordinate z.

(ii) Verify that the spherical metric $\sigma = 2\,|dz|/(1+|z|^2)$ has constant curvature $+1$.

(iii) Show that the curvature of the metric $\rho = c\,|dz|/(1-|z|^2)$ on the unit disk is the constant $\kappa \equiv -4/c^2$.

(iv) With ρ as in (iii), let $L(r)$ be the ρ-perimeter of a circle of ρ-radius $r > 0$. Show that

$$L(r) = 2\pi r - \frac{\pi}{3}\kappa r^3 + O(r^5) \qquad \text{as } r \to 0,$$

so that

$$\kappa = -\frac{1}{3}\,\frac{d^2}{dr^2}\left(\frac{L(r)}{2\pi r}\right)\Big|_{r=0}.$$

In other words, κ measures the third-order deviation of $L(r)$ from the Euclidean perimeter $2\pi r$ when r is small.

(45) (*Ahlfors's version of Schwarz Lemma*) Let X be a Riemann surface equipped with a C^2 conformal metric ρ whose Gaussian curvature is everywhere bounded above by a negative constant $-B$. If $f : \mathbb{D} \to X$ is a holomorphic map, show that

$$\|f'(z)\| = \frac{\rho(f(z))\,|f'(z)|}{\rho_{\mathbb{D}}(z)} \le \frac{1}{\sqrt{B}} \qquad \text{for all } z \in \mathbb{D}.$$

You may find it helpful to first replace the hyperbolic metric $\rho_{\mathbb{D}}$ by the metric

$$\rho_r(z) = \frac{2r\,|dz|}{r^2 - |z|^2}$$

on the disk $|z| < r < 1$ and note that the ratio $\rho(f(z))|f'(z)|/\rho_r(z)$ tends to zero as $|z| \to r$, so it has to take its maximum value somewhere in the disk $|z| < r$.

(46) Let X be a Riemann surface which admits a C^2 conformal metric whose curvature is bounded above by a negative constant. Check that X cannot be conformally isomorphic to $\widehat{\mathbb{C}}$. On the other hand, use Ahlfors's version of Schwarz Lemma to show that any holomorphic map $\mathbb{C} \to X$ must be constant, so X cannot be a parabolic Riemann surface. Conclude that a Riemann surface admits such a metric iff it is hyperbolic.

(47) Let $U \subset \mathbb{C}$ be a hyperbolic domain. For $z \in U$, let $\delta(z)$ denote the Euclidean distance from z to ∂U. As usual, let $\rho_U(z)\,|dz|$ denote the hyperbolic metric in U.

(i) If U is simply-connected, show that

$$\frac{1}{2\delta(z)} \leq \rho_U(z) \leq \frac{2}{\delta(z)}$$

for all $z \in U$. Thus, $\rho_U(z)$ blows up near ∂U at the same rate as $1/\delta(z)$. You may find it helpful to use Schwarz Lemma for the upper bound and Koebe 1/4-Theorem for the lower bound.

(ii) In general, show that there is a constant $C > 0$ such that

$$\frac{C}{|\delta(z) \log \delta(z)|} \leq \rho_U(z) \leq \frac{2}{\delta(z)}$$

for all $z \in U$.

(48) Suppose $X \subsetneq Y$ are hyperbolic Riemann surfaces. For $z \in X$, let $\delta(z)$ be the ρ_Y-distance from z to $\partial X \subset Y$. Show that the inclusion map $\iota : X \to Y$ satisfies

$$\tanh\left(\frac{\delta(z)}{2}\right) \leq \|i'(z)\| = \frac{\rho_Y(z)}{\rho_X(z)} \leq C\,|\delta(z)\,\log \delta(z)|,$$

where $C > 0$ is a constant.

(49) Suppose $U \subsetneq \mathbb{C}$ is simply-connected, $z, w \in U$, and $d = \mathrm{dist}_U(z, w)$. Use classical Köebe Distortion Theorem to show that

$$e^{-2d} \leq \frac{\rho_U(z)}{\rho_U(w)} \leq e^{2d}.$$

(50) Prove the following general version of Köebe Distortion Theorem: Suppose $U \subsetneq \mathbb{C}$ is simply-connected and $\phi : U \to \mathbb{C}$ is univalent. Let $K \subset U$ be compact, with hyperbolic diameter d. Then

$$\sup\left\{\frac{|\phi'(z)|}{|\phi'(w)|} : z, w \in K\right\} \leq e^{4d}.$$

(51) Show that normality is a local property. More precisely, let X and Y be Riemann surfaces and $\mathcal{F} \subset \mathrm{Hol}(X, Y)$. Show that $\mathcal{F}$ is normal iff every $z \in X$ has a neighborhood U such that $\mathcal{F}|_U \subset \mathrm{Hol}(U, Y)$ is normal.

(52) Let X and Y be Riemann surfaces, Y be compact, and $\mathcal{F} \subset \mathrm{Hol}(X, Y)$. Show that $\mathcal{F}$ is normal iff for every compact set $K \subset X$ there exists a constant

$M(K) > 0$ such that

$$\sup_{f \in \mathcal{F}, z \in K} \|f'(z)\| \leq M(K).$$

Here the norm $\|f'(z)\|$ can be measured with respect to any smooth Riemannian metrics on X and Y.

(53) Let $f_n(z) = z + n$. Show that $\{f_n\}$, as a family of holomorphic maps $\mathbb{C} \to \mathbb{C}$, tends locally uniformly to infinity in $\mathbb{C}$. On the other hand, show that $\{f_n\}$, as a family of holomorphic maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, converges pointwise to the constant function $\widehat{\mathbb{C}} \to \{\infty\}$ but this convergence is not uniform (in other words, f_n does not converge in $\mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$).

(54) Suppose $A, B, C : X \to \widehat{\mathbb{C}}$ are holomorphic maps with distinct values at each point of X. Let $f_n : X \to \widehat{\mathbb{C}}$ be a sequence of holomorphic maps such that

$$f_n(z) \neq A(z) \quad f_n(z) \neq B(z) \quad f_n(z) \neq C(z)$$

for every $z \in X$ and every n. Show that $\{f_n\}$ is normal.

(55) (*Picard's Great Theorem*) Let $f : \mathbb{D}^* \to \mathbb{C}$ be holomorphic, with an essential singularity at 0. Apply Montel's Theorem to an appropriately defined family of holomorphic maps to show that $\mathbb{C} \smallsetminus f(\mathbb{D}^*)$ contains at most one point.

(56) This problem shows that non-constant proper holomorphic maps have a well-defined "mapping degree" (the number of preimages of a point counting multiplicities). Suppose $f : X \to Y$ is a non-constant, proper holomorphic map between Riemann surfaces.

(i) Let C be the set of critical points of f and $V = f(C)$ be the set of critical values. Verify that C, hence V, is discrete. Show that

$$f : X \smallsetminus f^{-1}(V) \to Y \smallsetminus V$$

is a covering map of some finite degree d. It follows in particular that $f(X) = Y$.

(ii) Show that for every $w \in Y$,

$$\sum_{z \in f^{-1}(w)} \deg(f, z) = d.$$

Here $\deg(f, z)$ denotes the local degree of f at z.

(iii) Can you find a proper holomorphic map of degree 2 with infinitely many critical points?

(57) (*Branched coverings*) This exercise develops a topological analogue for proper holomorphic maps. Let X and Y be two connected oriented topological surfaces. A continuous map $f : X \to Y$ is called a *branched covering* if for every

$q \in Y$ there is an open ball $V(q)$ such that $f^{-1}(V(q))$ is the disjoint union of open balls $\bigcup_{p \in f^{-1}(q)} U(p)$, and the action $f : U(p) \to V(q)$ is like a power, that is, there are homeomorphisms $\varphi : U(p) \xrightarrow{\cong} \mathbb{D}$ and $\psi : V(q) \xrightarrow{\cong} \mathbb{D}$ such that $\psi \circ f \circ \varphi^{-1}(z) = z^k$ for some $k \geq 1$. The integer k, which is independent of the choice of φ and ψ, is called the *local degree* of f at p and is denoted by $\deg(f, p)$. We call $p \in X$ a *branch point* if $\deg(f, p) > 1$, and $q \in Y$ a *ramified point* if $q = f(p)$ for some branch point p. We denote by B and $R = f(B)$ the set of branch and ramified points of f, respectively.

(i) Show that B and R are discrete sets.

(ii) Show that $f : X \smallsetminus f^{-1}(R) \to Y \smallsetminus R$ is a covering map, with a well-defined degree $1 \leq d \leq +\infty$.

(iii) Assume $d < +\infty$. Show that for every $q \in Y$,

$$\sum_{p \in f^{-1}(q)} \deg(f, p) = d.$$

(iv) Prove that a non-constant holomorphic map between Riemann surfaces is a finite-degree branched covering iff it is proper. In this case, the branch points correspond to the critical points and the ramified points correspond to the critical values of f.

(58) Let $f : X \to Y$ be a branched covering of finite degree $d \geq 1$.

(i) Suppose $V \subset Y$ is a domain and $U \subset X$ is a connected component of $f^{-1}(V)$. Show that $f : U \to V$ is a branched covering of some degree $\leq d$ (in particular, it is surjective).

(ii) If in (i) the domain V is simply-connected and U contains no branch point, prove that $f : U \to V$ is a homeomorphism.

(59) (*The Monodromy Theorem*) Suppose $f : X \to Y$ is a non-constant proper holomorphic map, and $V \subset Y$ is a simply-connected domain containing no critical value of f. Then there exists a holomorphic map $g : V \to X$ which satisfies $f \circ g = \mathrm{id}$. One calls g a "holomorphic branch of f^{-1}."

(60) Let $f : \mathbb{D} \to \mathbb{D}$ be a degree two proper holomorphic map. Show that there exist $\sigma, \tau \in \mathrm{Aut}(\mathbb{D})$ such that $(\sigma \circ f \circ \tau)(z) = z^2$ for all z.

(61) Let X be a compact Riemann surface of genus $g \geq 1$ and $f : X \to X$ be a non-constant holomorphic map. Use Riemann-Hurwitz formula to show that

(i) If $g = 1$, then f is a covering map of finite degree (which could take any value ≥ 1).

(ii) If $g > 1$, then f is a biholomorphism.

(62) Let $f : X \to Y$ be a non-constant holomorphic map between compact hyperbolic surfaces. If f has $N \geq 1$ critical points counting multiplicities, show that

$$1 + \frac{N}{\chi(X)} < \sup_{z \in X} \|f'(z)\|^2 < 1.$$

(63) (*Degree-Genus Formula*) Let $X \subset \mathbb{CP}^2$ be a non-singular algebraic curve of degree d. Show that the genus of X is given by

$$g = \frac{(d-1)(d-2)}{2}$$

by completing the following sketch: Suppose X is defined as the zero locus of an irreducible degree d polynomial $P(z, w)$. Using the fact that $\mathbb{CP}^2 \smallsetminus \{\text{point}\}$ fibers over $\mathbb{CP}^1 \cong \widehat{\mathbb{C}}$, define a branched covering $X \to \widehat{\mathbb{C}}$ and apply Riemann-Hurwitz formula.

(64) Let $U, V \subset \widehat{\mathbb{C}}$ be hyperbolic domains, with V simply-connected. Suppose $f : U \to V$ is a proper holomorphic map of degree $d \geq 2$ with a single critical value v. Show that $f^{-1}(v)$ consists of a single point c, and there are bohilomorphisms $\varphi : U \smallsetminus \{c\} \to \mathbb{D}^*$ and $\psi : V \smallsetminus \{v\} \to \mathbb{D}^*$ such that $\psi \circ f \circ \varphi^{-1}(z) = z^d$ for all $z \in \mathbb{D}^*$.

(65) Suppose f is a holomorphic self-map of a hyperbolic Riemann surface X.

(i) If f has two fixed points in X, show that $f^{\circ n} = \mathrm{id}_X$ for some n.

(ii) If f has a fixed point z_0 whose multiplier $f'(z_0)$ is an n-th root of unity, show that $f^{\circ n} = \mathrm{id}_X$.

(66) Let X be a compact Riemann surface of genus $g \geq 2$. Show that every non-constant holomorphic map $X \to X$ is a finite-order automorphism. (A classical theorem of Hurwitz asserts that $\mathrm{Aut}(X)$ is a finite group of order at most $84(g-1)$.)

(67) Let X be a Riemann surface, $f : X \to X$ be holomorphic, $K \subset X$ be non-empty and compact, and $f(K) \subset K$. Prove that the following conditions are equivalent:

(i) There exists a conformal metric ρ defined in a neighborhood of K and a constant $\lambda > 1$ such that $\|f'(z)\|_\rho > \lambda$ for every $z \in K$.

(ii) There exists a conformal metric ρ defined in a neighborhood of K and constants $C > 0, \lambda > 1$ such that $\|(f^{\circ n})'(z)\|_\rho > C\,\lambda^n$ for every $z \in K$ and every $n \geq 1$.

(iii) There exists a conformal metric ρ defined in a neighborhood of K, a positive integer n and a constant $\lambda > 1$ such that $\|(f^{\circ n})'(z)\|_\rho > \lambda$ for every $z \in K$.

(Note that the metrics and the constants in (i), (ii), or (iii) are perhaps different). If any and hence all of these conditions are satisfied, we say that f is *expanding* on K. An important corollary is that when f is expanding on K and $z \in K$, $\|(f^{\circ n})'(z)\|_\rho \to +\infty$ exponentially fast for *any* conformal metric defined near K.

(68) For $\tau \in \mathbb{H}$, consider the lattice $\Lambda = \mathbb{Z} + \tau\,\mathbb{Z}$ in $\mathbb{C}$. Let g denote the affine map $z \mapsto \alpha z + \beta$, where $\alpha, \beta \in \mathbb{C}$.

(i) Show that g induces a holomorphic self-map of the torus $T = \mathbb{C}/\Lambda$ iff $\alpha\Lambda \subset \Lambda$ (note that this puts no restriction on β and is always satisfied if $\alpha \in \mathbb{Z}$). Show that the resulting map $T \to T$ is a covering map of degree $|\alpha|^2$; in particular, $|\alpha|^2$ must be an integer.

(ii) Show that there exists an $\alpha \notin \mathbb{Z}$ with the property $\alpha\Lambda \subset \Lambda$ iff τ satisfies a quadratic equation of the form

$$A\tau^2 + B\tau + C = 0 \quad \text{with} \quad A, B, C \in \mathbb{Z} \quad \text{and} \quad B^2 < 4AC.$$

Such a lattice is said to admit a *complex multiplication*.

(iii) Classify all lattices Λ and all numbers α such that

$$|\alpha| = 1, \quad \alpha \neq \pm 1 \quad \text{and} \quad \alpha\Lambda \subset \Lambda.$$

(iv) Fix any $\alpha \neq 0$ such that $\alpha\Lambda \subset \Lambda$. Verify that for any z_0, the equation $g(z) = \alpha z + \beta = z_0$ has $|\alpha|^2$ distinct roots in T. If $\alpha \neq 0, 1$, show that g has $|\alpha - 1|^2$ distinct fixed points in T; in particular, $|\alpha - 1|^2$ must also be an integer. More generally, if α is not an n-th root of unity, show that $g^{\circ n}$ has $|\alpha^n - 1|^2$ distinct fixed points in T; in particular, $|\alpha^n - 1|^2$ must be an integer.

(69) For $\tau \in \mathbb{H}$, consider the Weierstrass function $\wp_\tau : \mathbb{C} \to \widehat{\mathbb{C}}$ associated to the lattice $\mathbb{Z} + \tau\,\mathbb{Z}$. Let $f_\tau : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the degree n^2 Lattès map satisfying $f_\tau(\wp_\tau(z)) = \wp_\tau(nz)$ for all $z \in \mathbb{C}$.

(i) Show that for every pair $\tau, \sigma \in \mathbb{H}$, the Lattès maps f_τ and f_σ are topologically conjugate.

(ii) For what pairs $\tau, \sigma \in \mathbb{H}$ are f_τ and f_σ conformally conjugate?

(70) For a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, let $C(f)$ denote the set of critical points and $V(f) = f(C(f))$ the set of critical values of f. Show that for any $k \geq 1$,

$$C(f^{\circ k}) = \bigcup_{j=0}^{k-1} f^{-j}(C(f)) \quad \text{and} \quad V(f^{\circ k}) = \bigcup_{j=0}^{k-1} f^{\circ j}(V(f)).$$

(71) Let $f \in \mathrm{Rat}_d$ with $d \geq 2$. Suppose K is a non-empty compact set which is totally invariant under f (that is, $f^{-1}(K) = K$).

(i) If $\# K \leq 2$, show that K is contained in the exceptional set $\mathcal{E}(f)$ consisting of grand orbit finite points.

(ii) If $\# K \geq 3$, show that $K \supset J(f)$.

Part (ii) says that the Julia set is the smallest compact totally invariant set which has at least 3 points.

(72) Let $f \in \mathrm{Rat}_d$ with $d \geq 2$. If $\# \mathcal{E}(f) = 1$, show that f is conjugate to a polynomial. If $\# \mathcal{E}(f) = 2$, show that f is conjugate to $z \mapsto z^d$ or $z \mapsto z^{-d}$.

(73) Let $f, g \in \mathrm{Rat}_d$ with $d \geq 2$. If f and g commute, show that $J(f) = J(g)$. Is the converse necessarily true?

(74) Let $f \in \mathrm{Rat}_d$ with $d \geq 2$. Show that the *automorphism group* of f, defined by

$$\mathrm{Aut}(f) = \{\sigma \in \mathrm{Aut}(\widehat{\mathbb{C}}) : \sigma \circ f = f \circ \sigma\},$$

is always finite. Can you describe this group when $f(z) = z^d$?

(75) Let $f \in \mathrm{Rat}_d$ with $d \geq 2$, and suppose U is a Fatou component of f. Show that $f(U)$ is also a Fatou component and the map $f : U \to f(U)$ is proper with a well-defined degree $\leq d$.

(76) Let $f \in \mathrm{Rat}_d$ with $d \geq 2$. Suppose $U = f^{-1}(U)$ is a totally invariant Fatou component of f. Show that $\partial U = J(f)$ and all other Fatou components of f are simply-connected. For the second claim, you may find it useful to arrange $\infty \in U$ and apply the Maximum Principle.

(77) Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial map of degree $d \geq 2$. Define the basin of attraction of infinity by

$$A_\infty = \{z \in \mathbb{C} : f^{\circ n}(z) \to \infty \text{ as } n \to \infty\}.$$

Show that $A_\infty \subset F(f)$ is open, contains $\{z : |z| > R\}$ for a large $R > 0$, and $f^{-1}(A_\infty) = A_\infty$. Show that A_∞ is in fact a connected component of $F(f)$. Conclude that $J = \partial A_\infty$ and all other Fatou components of f (i.e., the bounded ones) are simply-connected.

(78) Let $f \in \mathrm{Rat}_d$ with $d \geq 2$. Suppose f has only a finite number k of Fatou components. Show that $k \leq 2$.

(79) (*Finite Blaschke products*) This problem discusses a special class of rational maps whose behavior is reminiscent of the simple maps $z \mapsto z^d$. Let $f : \mathbb{D} \to \mathbb{D}$ be a proper holomorphic map of degree d.

(i) Show that f is a finite Blaschke product, that is,

$$f(z) = \lambda \prod_{j=1}^{d} \left(\frac{z - a_j}{1 - \overline{a_j} z} \right),$$

for some $\lambda \in \partial\mathbb{D}$ and (not necessarily distinct) $a_1, \ldots, a_d \in \mathbb{D}$. In particular, f extends to a degree d rational map of the sphere.

(ii) Show that this rational map f commutes with the reflection $z \mapsto 1/\overline{z}$. Show that z is a critical point of f iff $1/\overline{z}$ is critical. Check that f has $d-1$ critical points in $\mathbb{D}$ and hence the same number of critical points in $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$. Conclude that f has no critical point on the unit circle.

(iii) Now let $d \geq 2$. Show that either $J(f) = \partial\mathbb{D}$ or $J(f)$ is a Cantor set in $\partial\mathbb{D}$. Give examples showing either case is possible. Conclude that f has one or two Fatou components. What are the possible types of these components?

(iv) Still assuming $d \geq 2$, suppose that $f(0) = 0$. Use Schwarz Lemma to show that every orbit in $\mathbb{D}$ tends to 0, and hence every orbit in $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ tends to ∞. Conclude that $J(f) = \partial\mathbb{D}$.

(v) Let f be as in (iv), so that $J(f) = \partial\mathbb{D}$. Show that $f : \partial\mathbb{D} \to \partial\mathbb{D}$ is a d-to-1 covering map which expands the Euclidean metric. More precisely, let $1 \leq m \leq d$ be the local degree of f at 0. Show that when $|z| = 1$, $|f'(z)| = m$ if $m = d$, and $|f'(z)| > m$ if $1 \leq m \leq d-1$.

(vi) Let f be as in (iv). Show that the action of the expanding map f on $\partial\mathbb{D}$ preserves Lebesgue measure on the circle and is ergodic with respect to it. (Think of the harmonic extension of a suitable function defined on the circle.)

(80) (*Shrinking Lemma*) Let $f \in \mathrm{Rat}_d$ with $d \geq 2$. Suppose $U \subset \widehat{\mathbb{C}}$ is a domain on which a branch g_n of f^{-n} can be defined for every $n \geq 1$. If $U \cap J(f) \neq \emptyset$, show that $\|g_n'\|_\sigma \to 0$ locally uniformly in U. Conclude that for every compact set $K \subset U$, the spherical diameter of $g_n(K)$ tends to zero as $n \to \infty$.

(81) Use Shrinking Lemma to give another proof for the fact that indifferent points in the Julia set as well as boundaries of Siegel disks and Herman rings are contained in the postcritical set.

(82) Recall that the action of a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is *ergodic* if $f^{-1}(E) = E$ implies E or $\widehat{\mathbb{C}} \smallsetminus E$ has measure zero. Use the classification of Fatou components to prove that $J(f) = \widehat{\mathbb{C}}$ if f is ergodic.

(83) Recall that if $f \in \mathrm{Rat}_d$ with $d \geq 2$, either $J(f) = \widehat{\mathbb{C}}$ and the action of f on $J(f)$ is ergodic, or $\mathrm{dist}_\sigma(f^{\circ n}(z), P(f)) \to 0$ for almost every $z \in J(f)$ (this is "Ergodic or Attracting Theorem").

(i) Show that if f is expanding, then $J(f)$ has measure zero.

(ii) Let us say that f is *geometrically finite* if every critical point in $J(f)$ has finite orbit (and hence all the critical points in the Fatou set tend to attracting or parabolic cycles). Generalizing (i), show that if f is geometrically finite, then either $J(f) = \widehat{\mathbb{C}}$ or $J(f)$ has measure zero.

(84) Let $f \in \mathrm{Rat}_d$ with $d \geq 2$. Suppose all the critical points of f are *strictly preperiodic*, i.e., they are not periodic but eventually map to periodic points. Show that $J(f) = \widehat{\mathbb{C}}$. As an example, $f(z) = (i/2)(z + z^{-1})$ has critical points at $z = \pm 1$ with critical orbits

$$\pm 1 \mapsto \pm i \mapsto 0 \mapsto \infty \circlearrowright$$

so $J(f)$ must be the entire sphere.

(85) Give an example of a non-identity homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ which satisfies $\partial\varphi = 1$ and $\overline{\partial}\varphi = 0$ almost everywhere.

(86) Show that a homeomorphism $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal iff there is a $C > 0$ such that for every quadruple (a, b, c, d) of distinct points in $\widehat{\mathbb{C}}$,

$$\mathrm{dist}(\chi(a, b, c, d), \chi(\varphi(a), \varphi(b), \varphi(c), \varphi(d))) \leq C.$$

Here "dist" is the hyperbolic distance in $\widehat{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$, and χ is the cross-ratio. Show that the maximal dilatation K of φ and the constant C depend only on each other.

(87) Let X be a Riemann surface and G be a finite group of quasiconformal homeomorphisms $X \to X$. Show that there exists a Riemann surface Y and a quasiconformal map $\varphi : X \to Y$ such that the conjugate group $\varphi \circ G \circ \varphi^{-1}$ is a subgroup of $\mathrm{Aut}(Y)$. In other words, a finite group of quasiconformal maps acts as a group of conformal maps if seen in an appropriate coordinate system. (This result can be generalized to any group of uniformly K-quasiconformal homeomorphisms.)

(88) Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a finite order diffeomorphism. Show that there exists a diffeomorphism $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ such that the conjugate map $\varphi \circ f \circ \varphi^{-1}$ is a rigid rotation.

(89) Let μ be a Beltrami differential on $\widehat{\mathbb{C}}$ with $\|\mu\|_\infty < 1$. Assume μ is symmetric with respect to the unit circle $\partial\mathbb{D}$, i.e., $\mu = I^*\mu$, where $I(z) = 1/\overline{z}$. If $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the unique quasiconformal map which fixes $0, 1, \infty$ and satisfies $\overline{\partial}\varphi = \mu\,\partial\varphi$, show that φ commutes with I so that $\varphi(\partial\mathbb{D}) = \partial\mathbb{D}$.

(90) Show that locally bounded Beltrami differentials are integrable. More precisely, let X be a Riemann surface and μ be a measurable Beltrami differential on X with the property that every point in X has a neighborhood U such that $\|\mu\|_U < 1$. Prove that there exists a Riemann surface Y and a locally quasiconformal homeomorphism $\varphi : X \to Y$ such that $\overline{\partial}\varphi = \mu\,\partial\varphi$. Moreover, φ is unique up to postcomposition with a biholomorphism. (As a special case, it follows that every continuous Beltrami differential is integrable.)

(91) Let μ be a continuous Beltrami differential on $\mathbb{D}$ which is rotationally symmetric, i.e., $R_\theta^*\mu = \mu$ for every rotation $R_\theta(z) = e^{i\theta}z$.

   (i) Show that there is a locally quasiconformal homeomorphism $\varphi : \mathbb{D} \to \mathbb{D}$ or $\mathbb{C}$, satisfying $\overline{\partial}\varphi = \mu\,\partial\varphi$, which commutes with every rotation (so that it has the form $\varphi(re^{it}) = \varphi(r)e^{it}$ in polar coordinates). You can prove this statement using the general theory, but it is instructive to verify it by solving the Beltrami equation for φ directly.

   (ii) As usual, let $D = (1 + |\mu|)(1 - |\mu|)$ be the real dilatation of μ, which by symmetry depends only on $r = |z|$. Show that the Riemann surface $(\mathbb{D}, \mu)$ is conformally isomorphic to $\mathbb{D}$ or $\mathbb{C}$ according as

$$\lim_{A \to 1^-} \int_{0.5}^{A} \frac{D(r)}{r}\, dr$$

   converges or diverges.