BIACCESSIBILITY IN QUADRATIC JULIA SETS

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Abstract. This paper consists of two nearly independent parts, both of which discuss the common theme of biaccessible points in the Julia set J of a quadratic polynomial $f : z \mapsto z^2 + c$.

In Part I, we assume that J is locally-connected. We prove that the Brolin measure of the set of biaccessible points (through the basin of attraction of infinity) in J is zero except when $f(z) = z^2 - 2$ is the Chebyshev map for which the corresponding measure is one. As a corollary, we show that a locally-connected quadratic Julia set is not a countable union of embedded arcs unless it is a straight line or a Jordan curve.

In Part II, we assume that f has an irrationally indifferent fixed point α. If z is a biaccessible point in J, we prove that the orbit of z eventually hits the critical point of f in the Siegel case, and the fixed point α in the Cremer case. As a corollary, it follows that the set of biaccessible points in J has Brolin measure zero.

Part I: The Locally-Connected Case

1.1. **Introduction.** Let $f : z \mapsto z^2 + c$ be a quadratic polynomial in the complex plane $\mathbb{C}$. Recall that the *filled Julia set* of f is

$$K = \{z \in \mathbb{C} : \text{The orbit } \{f^{\circ n}(z)\}_{n \geq 0} \text{ is bounded}\}$$

and the *Julia set* of f is the topological boundary of the filled Julia set:

$$J = \partial K.$$

Both sets are nonempty and compact, and the filled Julia set is full, i.e., the complement $\mathbb{C} \smallsetminus K$ is connected. Let $\psi : \overline{\mathbb{C}} \smallsetminus \overline{\mathbb{D}} \xrightarrow{\simeq} \overline{\mathbb{C}} \smallsetminus K$ be the unique conformal isomorphism, normalized as $\psi(\infty) = \infty$ and $\psi'(\infty) = 1$, which conjugates the squaring map to f:

$$\psi(z^2) = f(\psi(z)). \tag{1.1}$$

(The inverse ψ^{-1} is often called the *Böttcher* coordinate.) By the *external ray* R_t we mean the image of the radial line $\{\psi(re^{2\pi i t}) : r > 1\}$, where $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the *angle* of the ray. We say that R_t *lands* at $z \in J$ if $\lim_{r \to 1} \psi(re^{2\pi i t}) = z$. A point $z \in J$ is called *accessible* if there exists a simple arc in $\mathbb{C} \smallsetminus K$ which starts at infinity and terminates at z. According to a theorem of Lindelöf (see for example [**Ru**], Theorem 12.10), z is accessible exactly when there exists an external ray landing at z. We call z *biaccessible* if it is accessible through at least two distinct external rays. By a theorem of F. and M. Riesz [**Mi1**], $K \smallsetminus \{z\}$ is disconnected whenever z is biaccessible. It is interesting that the converse is also true. More precisely, if there are at least $n > 1$ connected components of $K \smallsetminus \{z\}$, then at least n distinct external rays land at z (see for example [**Mc**], Theorem 6.6).

Let us denote by $\gamma(t)$ the radial limit $\lim_{r \to 1} \psi(re^{2\pi i t})$. According to a classical theorem of Fatou (see [**Ru**], Theorem 11.32), $\gamma(t)$ exists for almost every $t \in \mathbb{T}$ in the sense of the Lebesgue measure. For all such angles t, it follows from (1.1) that γ conjugates the doubling map to the action of f on the Julia set:

$$\gamma(2t) = f(\gamma(t)). \tag{1.2}$$

When K, or equivalently J, is locally-connected, it follows from the theorem of Carathéodory (see [**Po**], Theorem 2.6) that γ is defined and continuous on the whole circle. In this case, the surjective map $\gamma : \mathbb{T} \to J$ is called the *Carathéodory loop*. Evidently the biaccessible points in J correspond to the points where γ fails to be one-to-one.

Whether or not J is locally-connected, the Lebesgue measure on the circle $\mathbb{T}$ pushes forward by γ to a probability measure μ on the Julia set. Complex analysts call μ the *harmonic measure* on J, but in the context of holomorphic dynamics, μ is called the *Brolin measure*. It has the following nice properties:

(i) The support of μ is the whole Julia set, with $\mu(J) = 1$.
(ii) μ is invariant under the $180°$ rotation $z \mapsto -z$, i.e., $\mu(-A) = \mu(A)$ for every measurable set $A \subset J$.
(iii) μ is f-invariant, i.e., $\mu(f^{-1}(A)) = \mu(A)$ for every measurable set $A \subset J$.
(iv) μ is ergodic in the sense that for every measurable set $A \subset J$ with $f^{-1}(A) = A$, we have $\mu(A) = 0$ or $\mu(A) = 1$.

All of these properties are immediate consequences of the corresponding properties of the Lebesgue measure and the angle-doubling map on the unit circle. Properties (ii) and (iii) are equivalent to the next property, which will be used repeatedly in this paper:

(v) $\mu(f(A)) = 2\mu(A)$ for every measurable set $A \subset J$ for which the restriction $f|_A$ is one-to-one.

Brolin proved that with respect to the measure μ the backward orbits of typical points have an asymptotically uniform distribution [**Br**]. Lyubich has proved that μ is the unique measure of maximal entropy $\log 2$. He has also constructed such invariant measures of maximal entropy for arbitrary rational maps of the Riemann sphere [**Ly**].

For $z \in J$, let $v(z)$ denote the number of external rays which land at z. (In Milnor's terminology [**Mi2**], this is called the *valence* of z.) For $0 \leq n \leq \infty$ define the measurable set $J_n = \{z \in J : v(z) = n\}$. It follows from elementary plane topology that the union $\bigcup_{n \geq 3} J_n$ is at most countable (see [**Po**], Proposition 2.18). On the other hand, the fact that almost every external ray (with respect to the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$) lands shows that $\mu(J_0) = 0$. Putting these two facts together, we conclude that $J = J_1 \cup J_2$ up to a set of μ-measure zero. Note that $v(f(z)) = v(z)$ unless z is the critical point. Therefore, if we neglect the grand orbit of the critical point which has μ-measure zero, it follows that both J_1 and J_2 must be f-invariant subsets of the Julia set. Ergodicity of μ then shows that up to a set of μ-measure zero, either $J = J_1$ or $J = J_2$.

As an example, for the *Chebyshev polynomial* $z \mapsto z^2 - 2$, the Julia set is the closed interval $[-2, 2]$ on the real line. Here every point is the landing point of exactly two rays except for the endpoints ± 2 where unique rays land, so $J = J_2$ is the case. There are no other known examples of quadratic Julia sets with two rays landing at almost every point. In fact, as I heard from J. Hubbard and later M. Lyubich, it is conjectured that a polynomial Julia set has this property *only if* it is a straight line segment in which case the map is conjugate to a Chebyshev polynomial, up to sign. In Part I of this paper, we will confirm this conjecture for quadratic Julia sets which are locally-connected. Part II, which is an expanded version of [**S-Z**], considers the Julia sets of quadratic polynomials with irrationally indifferent fixed points. By a completely different method we prove the sharper statement that every biaccessible point in J eventually maps to the critical point in the Siegel case and to the Cremer fixed point otherwise. As a byproduct, it follows that the set of biaccessible points in the Julia set has Brolin measure zero.

Addendum (June 1998). After the first version of this paper was circulated as Stony Brook IMS preprint in January of 1998, two successful attempts were made in order to settle the above conjecture in its full generality. Jan Kiwi has produced independently a combinatorial argument in the context of laminations, which is conceptually closer to the ideas of this paper. Stas Smirnov has a completely different approach based on A. Beurling's classical estimate for harmonic measures and A. Zdunik's dichotomy. Both methods prove a generalization of Theorem 1 of Part I of the present paper for connected polynomial Julia sets.

Acknowledgments. I am indebted to John Milnor who generously shared his ideas with me, which play an important role in both parts of the present paper. In particular, he suggested the idea of working with wakes, which simplified and unified part of the proof of Theorem 5 of Part II for both the Siegel and Cremer cases. Part II is an expanded version of a joint note with Dierk Schleicher which was written during my visit to the Technical University of München in July 1997, and covered the Cremer case with a rather different proof [**S-Z**]. I would like to thank him for very useful conversations which led to the joint work, which I have benefited from in Part II as well.

1.2. **Basic Definitions.** Let $f : z \mapsto z^2 + c$ be a quadratic polynomial whose filled Julia set K is locally-connected. f has two fixed points $(1 \pm \sqrt{1-4c})/2$ which are distinct if and only if $c \neq 1/4$. If $c \notin [1/4, \infty)$, so that the two fixed points have distinct real parts, then by convention the fixed point which is further to the left is called α and the other fixed point $1-\alpha$ is called β. If α is attracting or $\alpha = \beta$ ($\Leftrightarrow c = 1/4$), the Julia set of f is a Jordan curve with a unique external ray landing at every point. Hence there are no biaccessible points at all and Theorem 1 below is trivially true. *So we may assume that $\alpha \neq \beta$ and α is not attracting.* It follows that either $\alpha \in J$, or else α is the center of a fixed Siegel disk for f.

By an *embedded arc* in K we mean any subset of K homeomorphic to the closed interval $[0,1] \subset \mathbb{R}$. Since K is locally-connected, for any two points $x, y \in K$ there exists an embedded arc η in K which connects x to y. If K has no interior so that $J = K$ is full, then η is uniquely determined by the two endpoints x and y. If K does have interior, however, there is usually more than one choice for η. In what follows, we will show how to choose a canonical embedded arc between any two points in the filled Julia set.

Suppose that *int*(K) is non-vacuous. Every component U of this interior is a bounded Fatou component whose closure $\overline{U}$ is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$ since K is locally-connected. According to Fatou and Sullivan (see for example [**Mi1**]), every such component eventually maps to a periodic Fatou component which is either the immediate basin of attraction of an attracting periodic point, or an attracting petal for a parabolic periodic point, or a periodic Siegel disk. We refer to these cases simply as *hyperbolic*, *parabolic* and *Siegel* cases. Note that in the hyperbolic and parabolic cases the critical point 0 belongs to a central Fatou component which we denote by U_0. Also by our assumption on the α-fixed point, periodic Fatou components in the hyperbolic and parabolic cases form a cycle of period > 1.

Next, we would like to choose a *center* $c(U)$ in every bounded Fatou component U subject only to the following conditions:

- **(C1)** $c(-U) = -c(U)$,
- **(C2)** If U contains the critical value $c = f(0)$, then $c(U) = c$,
- **(C3)** If U contains the fixed point α, then $c(U) = \alpha$.

If follows from **(C1)** that whenever the critical point 0 belongs to the Fatou set, it is the center of the corresponding Fatou component U_0: $c(U_0) = 0$. Also **(C3)** corresponds to the case where the α-fixed point is the center of a fixed Siegel disk U.

Given any bounded Fatou component U, there exists a homeomorphism $\phi : \overline{U} \xrightarrow{\simeq} \overline{\mathbb{D}}$ which is holomorphic in U with $\phi(c(U)) = 0$. An arc in $\overline{U}$ of the form $\phi^{-1}\{re^{i\theta} : 0 \leq a \leq r \leq b \leq 1\}$ is called a *radial arc*. Since ϕ is unique up to post-composition with a rigid rotation of $\overline{\mathbb{D}}$, radial arcs in $\overline{U}$ are well-defined.

Following [**D-H**], we call an embedded arc I in K *regulated* if for every bounded Fatou component U, the intersection $I \cap \overline{U}$ is either empty or a point or consists of radial arcs in $\overline{U}$ (see also [**Do3**], where he uses the word "legal" for regulated).

Lemma 1. *Given any two points $x, y \in K$, there exists a unique regulated arc I in K with endpoints x, y. Furthermore, if η is any embedded arc in K which connects x to y, then $I \cap J \subset \eta \cap J$.*

Proof. Take any embedded arc η in K with endpoints x, y. It is easy to see how one can deform η to a regulated arc I. Let U be a bounded Fatou component whose closure intersects η. Choose any parametrization $h : [0,1] \to K$ with $\eta = h([0,1])$, and define

$$t_0 = \inf\{t \in [0,1] : h(t) \in \overline{U}\},$$
$$t_1 = \sup\{t \in [0,1] : h(t) \in \overline{U}\}.$$

In other words, t_0 is the first moment η hits $\overline{U}$ and t_1 is the last moment η stays in $\overline{U}$. If $t_0 \neq t_1$, replace the sub-arc of η from $h(t_0)$ to $h(t_1)$ by the radial arc from $h(t_0)$ to $c(U)$ followed by the radial arc from $c(U)$ to $h(t_1)$ (see Fig. 1). If $h(t_0)$ and $h(t_1)$ happen to be on the same radial arc, simply connect the two by the radial arc between them.

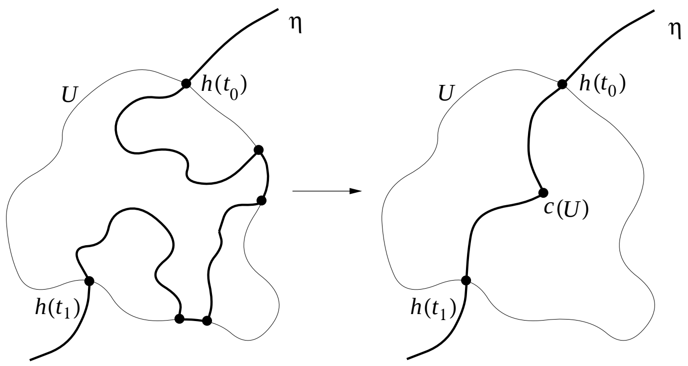

Figure 1. Deforming an embedded arc into a regulated arc.

Applying this construction to the intersection with every such Fatou component, we obtain a regulated arc I with endpoints x, y. Evidently we have the inclusion $I \cap J \subset \eta \cap J$.

To prove uniqueness, suppose that I and I' are both regulated, with the same endpoints x, y. If $I \neq I'$, then the complement $\mathbb{C} \smallsetminus (I \cup I')$ has a bounded connected component V. By the Maximum Principle, V is contained in some bounded Fatou component U. It follows that the boundary ∂V must be contained in a union of at most four radial arcs in $\overline{U}$. But a finite union of radial arcs cannot bound an open set in $\overline{U}$. Therefore, $I = I'$. □

The regulated arc I given by the above lemma is denoted by $[x, y]$. The open arc (x, y) is defined by $[x, y] \smallsetminus \{x, y\}$, and similarly we can define the semi-open arc $[x, y)$.

More generally, given finitely many points $x, y, \ldots, z$ in K, there is a unique smallest connected set $[x, y, \ldots, z] \subset K$ made up of regulated arcs which contains all of these points. In fact this set is always a (finite) topological tree. We call $[x, y, \ldots, z]$ the *regulated tree* generated by $\{x, y, \ldots, z\}$. A vertex of this tree with exactly one edge attached to it is called an *end* of the tree. A point which is not an end is called an *interior point* of the tree. It follows easily from **(C1)** that

$$[-x, -y, \ldots, -z] = -[x, y, \ldots, z] \tag{1.3}$$

In the case of three distinct points, $[x, y, z]$ is either homeomorphic to a closed interval or to a letter Y. The first case occurs if and only if one of the points belongs to the regulated arc connecting the other two. In the second case, the three points x, y, z are ends of the tree $[x, y, z]$. In other words, there is a unique interior point $p \in [x, y, z]$ such that $[x, p] \cap [y, p] = [x, p] \cap [z, p] = [y, p] \cap [z, p] = \{p\}$ (see Fig. 2). In this case, we call $[x, y, z]$ a *tripod*. Point p is called the *joint* of this tripod.

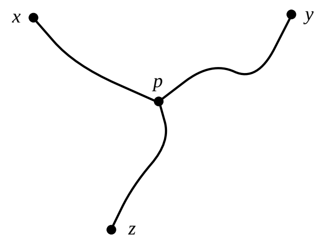

FIGURE 2. A tripod $[x, y, z]$ with joint p.

The regulated trees as defined above are not preserved by the dynamics of f. In fact, when K has interior, the center of a bounded Fatou component U is not necessarily mapped by f to that of $f(U)$. Hence regulated arcs in U do not map to regulated arcs in $f(U)$. This difficulty can be most conveniently overcome by deforming the polynomial f rel the Julia set into a new map F which respects the centers. To this end, it suffices to note that for every bounded Fatou component U, there is a homeomorphism between $\overline{U}$ and the cone over ∂U which sends $c(U)$ to the cone point and restricts to the identity map on ∂U. We can define F so as to preserve this cone structure on various bounded Fatou components. For example, for any component U and any $p \in \partial U$ take the Poincaré geodesic in U between $c(U)$ and p and define $F : \overline{U} \to f(\overline{U})$ so as to map this geodesic isometrically to the unique Poincaré geodesic between $c(f(U))$ and $f(p) \in \partial f(U)$. (Note that by our assumption $f(U) \neq U$ unless U is a fixed Siegel disk for which the α fixed point is the center. So in any case α is still a fixed point of F.) Apply this construction to every bounded Fatou component and let $F = f$ anywhere else. The map F will be the required modification of f which satisfies the following properties:

- **(F1)** $F(c(U)) = c(F(U))$ for every bounded Fatou component U. In particular, by **(C2)**, whether or not the critical point 0 belongs to the Fatou set, $F(0) = f(0) = c$ is always the critical value of f.
- **(F2)** $F = f$ on the closure of the basin of attraction of infinity.

(F3) $F(z) = F(z') \Leftrightarrow z = \pm z'$.
(F4) α and β are the only fixed points of F.

Also, since the support of the Brolin measure is the Julia set where f and F agree, it follows that properties (iii) and (v) in section 1.1 also hold for F. In other words,

(F5) $\mu(F^{-1}(A)) = \mu(A)$ for any measurable set $A \subset \mathbb{C}$, and
(F6) $\mu(F(A)) = 2\mu(A)$ for any measurable set $A \subset \mathbb{C}$ for which $F|_A$ is one-to-one.

Lemma 2. *Let $x, y, \ldots, z \in K$. Suppose that the critical point 0 is not an interior point of the tree $[x, y, \ldots, z]$. Then F maps $[x, y, \ldots, z]$ homeomorphically to $[F(x), F(y), \ldots, F(z)]$.*

In this case, we simply write

$$F : [x, y, \ldots, z] \xrightarrow{\simeq} [F(x), F(y), \ldots, F(z)].$$

Proof. First let us show that F restricted to $[x, y, \ldots, z]$ is injective. If not, it follows from **(F3)** that $[x, y, \ldots, z]$ contains a pair $\pm a$ of symmetric points. By (1.3), we see that $[a, -a] = -[a, -a]$. Hence the 180° rotation from the arc $[a, -a]$ to itself must have a fixed point, namely the critical point 0. But this implies that 0 is an interior point of $[x, y, \ldots, z]$, contrary to our assumption.

Therefore, F restricted to $[x, y, \ldots, z]$ is injective. The image tree $F([x, y, \ldots, z])$ is evidently connected and contains all of the image points $F(x), F(y), \ldots, F(z)$. Since all the ends of $F([x, y, \ldots, z])$ are among $F(x), F(y), \ldots, F(z)$, we conclude that it is also minimal. To finish the proof, it is enough to show that the image of every regulated arc in $[x, y, \ldots, z]$ is a regulated arc. But this follows from **(F1)** since F preserves the centers hence the radial arcs in bounded Fatou components of f. □

Definition 1. By the *spine* of the filled Julia set K we mean the unique regulated arc $[-\beta, \beta]$ between the β-fixed point and its preimage $-\beta$, which are the landing points of the unique external rays R_0 and $R_{1/2}$ respectively. By (1.3), the spine is invariant under the 180° rotation $z \mapsto -z$. In particular, the critical point 0 always belongs to the spine.

Let $z \in J$ be a biaccessible point, with a ray pair (R_t, R_s) landing at z and $0 < t < s < 1$. If $z \notin [-\beta, \beta]$, it follows that both t and s satisfy $0 < t < s < 1/2$ or $1/2 < t < s < 1$. Consider the orbit of the ray pair (R_t, R_s) under f. Since there exists an integer $n > 0$ such that $1/2 \leq 2^n s - 2^n t < 1$, the corresponding rays $f^{\circ n}(R_t)$ and $f^{\circ n}(R_s)$ must belong to different sides of the curve $R_{1/2} \cup [-\beta, \beta] \cup R_0$ (see Fig. 3). Therefore $f^{\circ n}(z) \in [-\beta, \beta]$. This means that the set B of all biaccessible points in the Julia set is contained in the union of preimages of the spine:

$$B \subset \bigcup_{n \geq 0} f^{-n}[-\beta, \beta]. \tag{1.4}$$

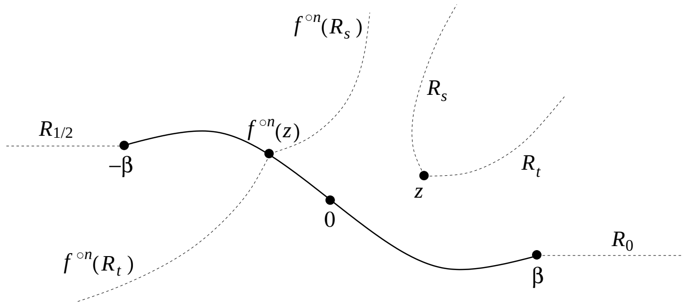

FIGURE 3

1.3. **Main Theorem and Supporting Lemmas.** Our main goal in Part I is to prove the following result:

Theorem 1. *If the Julia set J of the quadratic polynomial $f : z \mapsto z^2 + c$ is locally-connected, then the set of all biaccessible points in J has Brolin measure zero unless f is the Chebyshev polynomial $z \mapsto z^2 - 2$ for which the corresponding measure is one.*

By (1.4), it suffices to show that for every non-Chebyshev quadratic, the Brolin measure $\mu[-\beta, \beta]$ of the spine is zero.

The proof depends on several lemmas which will be given in this section and section 1.4 below. Some of these lemmas are of independent interest in studying the combinatorial structure of quadratic Julia sets. Unless otherwise stated, the Julia set J is assumed to be locally-connected.

Lemma 3.

(a) *Any point in the Julia set J which belongs to the boundary of two Fatou components is necessarily biaccessible.*
(b) *Let η be any embedded arc in the filled Julia set K and z be a point in $\eta \cap J$ which is not an endpoint of η. Then either z is biaccessible or it belongs to the boundary of a unique bounded Fatou component.*

Proof. (a) Let U and U' be two Fatou components with $z \in \partial U \cap \partial U'$. Assume that z is not biaccessible. Then $K \smallsetminus \{z\}$ is connected, so there exists an embedded arc η in K between $c(U)$ and $c(U')$ which avoids z. By Lemma 1, $I \cap J \subset \eta \cap J$, where $I = [c(U), c(U')] = [c(U), z] \cup [z, c(U')]$ is the unique regulated arc between $c(U)$ and $c(U')$. It follows that η must contain z, which is a contradiction.

(b) If z is not biaccessible, then $K \smallsetminus \{z\}$ is connected. Hence there exists an embedded arc η' in K between the two endpoints of η which avoids z. Take a bounded connected component V of the complement $\mathbb{C} \smallsetminus (\eta \cup \eta')$ which contains z in its closure. By the Maximum Principle, V must be contained in a bounded Fatou component. Hence z belongs to the boundary of this bounded Fatou component. Uniqueness follows from part (a). $\square$

Corollary 1. *Let $f : z \mapsto z^2 + c$ have locally-connected Julia set. If the α-fixed point is not attracting and $\alpha \neq \beta$, then neither the β-fixed point nor any of its preimages can belong to the boundary of a bounded Fatou component of f.*

Proof. Assume there exists a bounded Fatou component U with $\beta \in \partial U$. Then $\beta \in \partial U \cap \partial f(U)$. If $U = f(U)$, it must be a fixed Siegel disk by the assumption. But in this case $f|_{\partial U}$ is conjugate to an irrational rotation so it cannot have a fixed point. Therefore $U \neq f(U)$. By Lemma 3(a), β will be biaccessible. But this is impossible since the β-fixed point is always the landing point of the unique ray R_0. □

Remark 1. In the non locally-connected case, it is *not* known if the β-fixed point can be on the boundary of any bounded Fatou component. In fact, it is not known if there are examples of quadratic polynomials with a fixed Siegel disk whose boundary is the whole Julia set. Any such quadratic would provide a counterexample to the above corollary in the non locally-connected case.

Lemma 4. *If $x \notin [-\beta, \beta]$, then $[-\beta, x, \beta]$ is a tripod.*

Proof. Otherwise, we must have $-\beta \in (x, \beta)$ or $\beta \in (x, -\beta)$. In either case, it follows that $-\beta$ or β belongs to the interior of an embedded arc in the filled Julia set. But β is the landing point of the unique ray R_0. Since the orbit $-\beta \mapsto \beta$ does not pass through the critical point, it follows that $-\beta$ is also the landing point of the unique ray $R_{1/2}$. By Lemma 3(b), either β or $-\beta$ must be on the boundary of a bounded Fatou component, which contradicts Corollary 1. □

Here is a definition which will be used repeatedly in all subsequent arguments:

Definition 2. We define a projection $\pi : K \to [-\beta, \beta]$ as follows: For $x \in [-\beta, \beta]$, let $\pi(x) = x$. If $x \notin [-\beta, \beta]$, then $[-\beta, x, \beta]$ is a tripod by Lemma 4, and we define $\pi(x) \in (-\beta, \beta)$ to be the joint of this tripod.

Note that $\pi(x)$ can be described as the unique point in $[-\beta, \beta]$ such that for any y on the spine, $[x, \pi(x)] \subset [x, y]$. Set theoretically π a retraction from K onto its spine. However, when K has interior, π is not continuous.

For simplicity, we denote the regulated arc $[x, \pi(x)]$ by I_x. Since $\pi(-x) = -\pi(x)$, we have $I_{-x} = -I_x$.

Lemma 5. *The α-fixed point belongs to $(-\beta, 0)$.*

Proof. First we prove that $\alpha \in (-\beta, \beta)$. In fact, if α belonged to J and were off the spine, then the external rays which land at α would all belong to one side of the curve $R_{1/2} \cup [-\beta, \beta] \cup R_0$. This would contradict the fact that the angle-doubling map on the circle has no forward orbit which is entirely contained in the interval $(0, 1/2)$ or $(1/2, 1)$. On the other hand, if α belonged to the Fatou set and were off the spine, then it would have to be the center of a fixed Siegel disk whose closure by **(C3)** touches $[-\beta, \beta]$ at the unique point 0. Take the external ray R_t which lands at the critical value c. Since the entire orbit of c is on one side of the curve $R_{1/2} \cup [-\beta, \beta] \cup R_0$, the forward orbit of t under the doubling map must be entirely contained in one of the intervals $(0, 1/2)$ or $(1/2, 1)$, which is again a contradiction. Therefore, $\alpha \in (-\beta, \beta)$.

Now suppose that $\alpha \in (0, \beta)$. Then $[\alpha, \beta] \subset (0, \beta]$. Hence $F : [\alpha, \beta] \xrightarrow{\simeq} [\alpha, \beta]$ by Lemma 2. By **(F4)**, there is no fixed point of F in (α, β). Suppose that $[\alpha, \beta] \subset J$. Then f repels all points in $[\alpha, \beta]$ close to α and β. Since $f = F$ on the Julia set, the same must be true for F. Hence there has to be an attracting fixed point for F somewhere in (α, β), which is a contradiction. Therefore $[\alpha, \beta]$ intersects a bounded Fatou component U. Passing to some iterate $f^{\circ n}(U) = F^{\circ n}(U)$, we may as well assume that U is periodic. Since F acts monotonically on $[\alpha, \beta]$, U must be fixed. Hence U is a Siegel disk with $c(U) = \alpha$. Now ∂U intersects $[\alpha, \beta]$ at a unique point p which is not the β-fixed point by Corollary 1. Clearly $F(p) = p$, which is a contradiction. This shows that $\alpha \in (-\beta, 0)$, and completes the proof. ☐

Lemma 6. *There exists an F-preimage ω of 0 in $(-\beta, \alpha)$. The other preimage $-\omega$ is then in $(-\alpha, \beta)$.*

Proof. $F : [-\beta, \alpha] \xrightarrow{\simeq} [\beta, \alpha]$ by Lemma 2 since $0 \notin (-\beta, \alpha)$ by Lemma 5. Again by Lemma 5 we have $0 \in (\beta, \alpha)$, which shows there exists a unique $\omega \in (-\beta, \alpha)$ with $F(\omega) = 0$. ☐

Fig. 4 shows the relative position of the points along the spine.

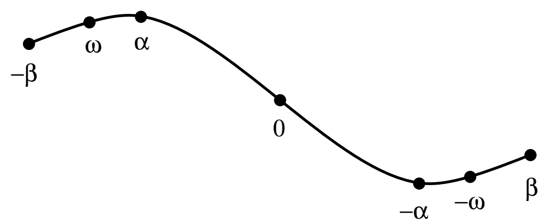

FIGURE 4

Lemma 7. *Let $c = f(0) = F(0)$ be the critical value. Then $\pi(c) \in [-\beta, \alpha]$. If $\pi(c) = -\beta$, then $c = -\beta$ in which case $f(z) = z^2 - 2$.*

Proof. By Lemma 2 we have $F : [0, \beta] \xrightarrow{\simeq} [c, \beta] = I_c \cup [\pi(c), \beta]$. Since $-\alpha \in [0, \beta]$, by **(F3)** and **(F4)** we must have $F(-\alpha) = \alpha \in [c, \beta]$. This is possible only if $\alpha \in [\pi(c), \beta]$, which is equivalent to $\pi(c) \in [-\beta, \alpha]$ (see Fig. 5).

If $\pi(c) = -\beta$, then $c = -\beta$ by Lemma 4. It is easy to see that $z \mapsto z^2 - 2$ is the only quadratic polynomial with the critical orbit $0 \mapsto c \mapsto \beta$. ☐

Lemma 8. *Suppose that f is not the Chebyshev polynomial. Let $f(\xi) = F(\xi) = -\beta$. Then ξ does not belong to the spine $[-\beta, \beta]$. Furthermore, $\pi(\xi) \in [-\alpha, \alpha]$ and*

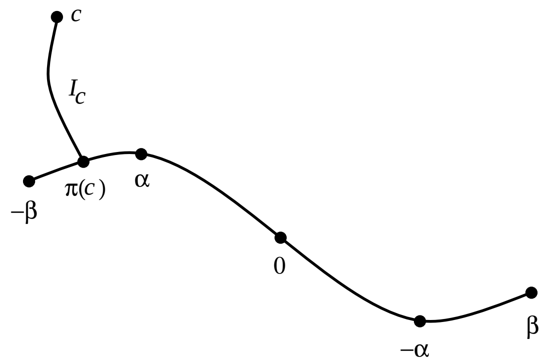

FIGURE 5

$F(\pi(\xi)) = \pi(c)$, with

$$c \in [-\beta, \beta] \Leftrightarrow \pi(\xi) = 0.$$

Proof. First suppose that $\pi(\xi) \neq 0$. Replacing ξ by $-\xi$ if necessary, we may assume that $\pi(\xi) \in (0, \beta)$. Then $F : [\xi, \beta] \xrightarrow{\simeq} [-\beta, \beta]$, hence $-\alpha \in [\xi, \beta]$ which implies that $-\alpha \in [\pi(\xi), \beta]$, or equivalently, $\pi(\xi) \in (0, -\alpha]$. Also, since $0 \notin [\xi, \beta]$, c cannot belong to the spine $[-\beta, \beta]$. By Lemma 7, $\pi(c) \in (-\beta, \alpha]$. By Lemma 2 the set $[\xi, 0, \beta]$ maps homeomorphically to the tripod $[-\beta, c, \beta]$, hence it must also be a tripod, with $\xi \notin [-\beta, \beta]$, and with the joint $\pi(\xi)$ mapped to $\pi(c)$ by F (see Fig. 6).

Now suppose that $\pi(\xi) = 0$. Then by a similar argument, the set $[\xi, 0, \beta] = [\xi, \beta]$ still maps homeomorphically to the spine $[-\beta, \beta]$ since it does not contain a pair of symmetric points about the origin. In particular, c must belong to the spine. By Lemma 7, $c = \pi(c) \in (-\beta, \alpha]$. $\square$

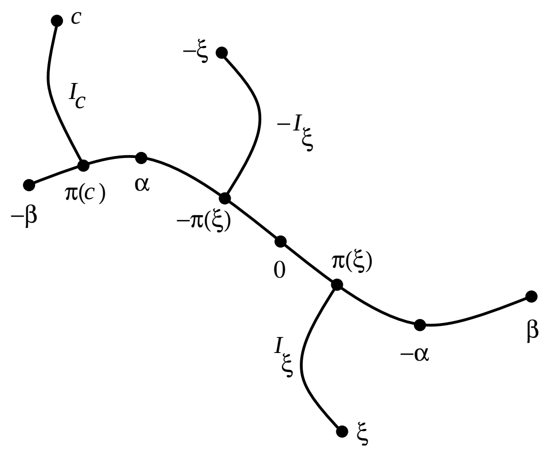

FIGURE 6

Corollary 2. *F maps $[0, \pm\pi(\xi)]$ to I_c and $\pm I_\xi$ to $[-\beta, \pi(c)]$ homeomorphically (see Fig. 6).*

Thus in all non-Chebyshev cases we have the situation illustrated in Fig. 6 (except that I_c may collapse to a point $\Leftrightarrow$ $[-\pi(\xi), \pi(\xi)]$ may collapse to a point, or alternatively $\pi(c)$ may coincide with α). Here

$$\pm\xi \overset{F}{\mapsto} -\beta \overset{F}{\mapsto} \beta,$$

$$\pm\pi(\xi) \overset{F}{\mapsto} \pi(c),$$

and

$$\pm\omega \overset{F}{\mapsto} 0 \overset{F}{\mapsto} c,$$

where ω lies somewhere between $-\beta$ and α.

Lemma 9. *Suppose that f is not the Chebyshev polynomial. Then the Brolin measure $\mu[-\beta, \beta]$ of the spine is zero if and only if $\mu(I_c) = 0$.*

Note that the condition $\mu(I_c) = 0$ is trivially satisfied if $c = \pi(c)$ belongs to the spine. The latter happens, for example, when the Julia set of $f(z) = z^2 + c$ with $c \in \mathbb{R}$ is full. When the Julia set is full, it is conjectured that the critical value belongs to the spine if and only if c is real.

Proof. By Lemma 8, for one preimage ξ of $-\beta$, we have $\pi(\xi) \in [0, -\alpha]$, and then the other preimage $-\xi$ satisfies $\pi(-\xi) \in [\alpha, 0]$. For simplicity, let $z_0 = \pi(\xi)$ and $z_n = F^{\circ n}(z_0)$. It follows from Corollary 2 that

$$F^{-1}([-\beta, \beta] \cup I_c) = [-\beta, \beta] \cup I_\xi \cup -I_\xi. \tag{1.5}$$

By property (ii) in section 1.1 and **(F5)**, we have

$$\mu(I_\xi) = \mu(-I_\xi) = \frac{1}{2}\mu(I_c). \tag{1.6}$$

Note that $z_1 = \pi(c) \in (-\beta, \alpha]$ by Lemma 8 and Lemma 7. By Corollary 2, **(F6)** and (1.6),

$$\mu[-\beta, z_1] = \mu(F(I_\xi)) = 2\mu(I_\xi) = \mu(I_c). \tag{1.7}$$

If $\mu[-\beta, \beta] = 0$, then $\mu[-\beta, z_1] = 0$, hence $\mu(I_c) = 0$ by (1.7). Conversely, if $\mu(I_c) = 0$, then $\mu[-\beta, z_1] = 0$. To prove $\mu[-\beta, \beta] = 0$, we distinguish two cases:

• *Case 1.* $z_1 \in [\omega, \alpha]$. Then $\mu[-\beta, \omega] \leq \mu[-\beta, z_1] = 0$. Hence $\mu[0, \beta] = 2\mu[-\beta, \omega] = 0$, which by symmetry implies $\mu[-\beta, \beta] = 0$.

• *Case 2.* $z_1 \in (-\beta, \omega)$. Then $z_2 = F(z_1) \in F(-\beta, \omega) = (0, \beta)$ and $\mu[z_2, \beta] = 2\mu[-\beta, z_1] = 0$. If $z_2 \in [0, -\omega]$, then $\mu[-\omega, \beta] = 0$ and it follows by an argument similar to *Case 1* that $\mu[-\beta, \beta] = 0$. So let us assume that $z_2 \in (-\omega, \beta)$. We can repeat the above argument by considering $z_3 = F(z_2) \in (0, \beta)$. If $z_3 \in [0, -\omega]$, we have $\mu[-\beta,\beta]=0$, otherwise $z_3\in(-\omega,\beta)$ and we continue. If this process never stops, it follows that $z_n\in(-\omega,\beta)$ and $(z_{n+1},\beta]\supset[z_n,\beta]$ for all n. The limit of the monotone sequence $\{z_n\}$ will then be a fixed point of F in $(-\omega,\beta)$, which contradicts **(F4)**. □

1.4. **The Proof.** The idea of the proof of Theorem 1 is as follows: We consider the n-th iterate of $c=f(0)=F(0)$, $c_n=F^{\circ n}(c)$. Under the assumption $\mu(I_c)>0$, we show that c_n cannot belong to the spine and the Brolin measure of the arc I_{c_n} tends to infinity as $n\to\infty$, which is clearly impossible since $\mu(J)=1$. Hence we must have $\mu(I_c)=0$. By Lemma 9, this proves the theorem.

Definition 3. Let I_1 and I_2 be two regulated arcs in the filled Julia set K. We say that I_1 and I_2 *overlap* if the intersection $I_1\cap I_2$ contains more than one point. It follows that $I_1\cap I_2$ is a non-degenerate regulated arc I in K. We often say that I_1 and I_2 *overlap along* I.

It is not hard to check that for $x,y\in K\smallsetminus[-\beta,\beta]$, the arcs I_x and I_y overlap if and only if x and y belong to the same connected component of $K\smallsetminus[-\beta,\beta]$. In particular, we must have $\pi(x)=\pi(y)$.

Lemma 10. *Let $x\in K\smallsetminus[-\beta,\beta]$. Then one and only one of the following cases occurs, as illustrated in Figures 7, 8, 9:*

(a) *I_x and I_ξ (or $-I_\xi$) overlap along an arc I_y. Then F maps $[x,y]$ homeomorphically to $I_{F(x)}=[F(x),F(y)]$.*
(b) *$\pi(x)\in(-\pi(\xi),\pi(\xi))$. Then F maps I_x homeomorphically to the arc $F(I_x)=[F(x),F(\pi(x))]$. In this case, $I_{F(x)}$ and I_c overlap along $I_{F(\pi(x))}=[F(\pi(x)),\pi(c)]$.*
(c) *$\pi(x)\notin(-\pi(\xi),\pi(\xi))$ and I_x and $\pm I_\xi$ do not overlap. Then F maps I_x homeomorphically to $I_{F(x)}$.*

Proof. (a) If $x\in I_\xi$ or $-I_\xi$, then $y=x$ and the result it trivial. Otherwise, $[\xi,x,\pi(x)=\pm\pi(\xi)]$ maps homeomorphically to $[-\beta,F(x),\pi(c)]$ (see Fig. 7). Hence $F(y)=\pi(F(x))$ and the result follows.

(b) If $\pi(x)\in(-\pi(\xi),\pi(\xi))$, then $F(\pi(x))\in I_c\smallsetminus\{\pi(c)\}$, hence $I_{F(x)}$ and I_c overlap along $I_{F(\pi(x))}$ (see Fig. 8).

(c) Since $\pi(x)\notin(-\pi(\xi),\pi(\xi))$, $F(\pi(x))\in[-\beta,\beta]$. So the claim is proved once we show that $\pi(F(x))=F(\pi(x))$. If these two points are distinct, then the non-degenerate arc $I=[\pi(F(x)),F(\pi(x))]\subset[-\beta,\beta]$ is contained in $[F(x),F(\pi(x))]$ (see Fig. 9). Hence $F^{-1}(I)$ will be a non-degenerate arc in $I_x\cap I_\xi$ or $I_x\cap -I_\xi$, which contradicts our assumption. □

Let us put $m=\mu(I_c)$. By (1.6), we have $\mu(\pm I_\xi)=m/2$.

Corollary 3. *If $x\in K\smallsetminus[-\beta,\beta]$ and $\mu(I_x)\geq 2m$, then $\mu(I_{F(x)})\geq\dfrac{3}{2}\mu(I_x)$.*

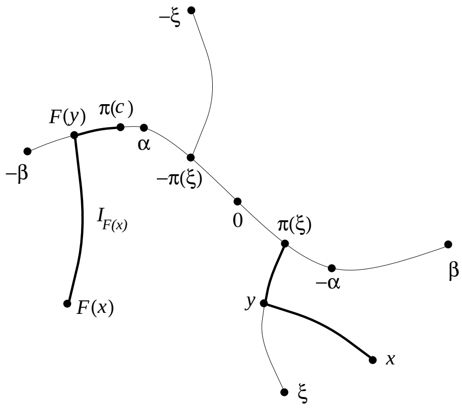

FIGURE 7

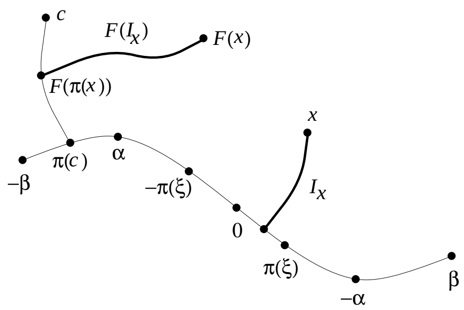

FIGURE 8

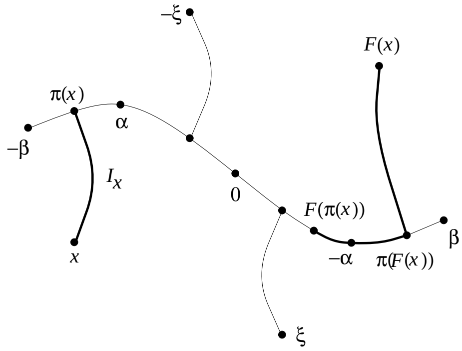

FIGURE 9

Proof. By Lemma 10 one and only one of the cases (a)-(c) occurs. In case (b), we have $\mu(I_{F(x)}) = \mu(F(I_x)) + \mu(I_{F(\pi(x))}) \geq \mu(F(I_x)) = 2\mu(I_x)$ and in case (c), $\mu(I_{F(x)}) = 2\mu(I_x)$. In case (a),

$$\begin{aligned} \mu(I_{F(x)}) &= \mu[F(x), F(y)] = 2\mu[x, y] \\ &= 2(\mu(I_x) - \mu(I_y)) \\ &\geq 2(\mu(I_x) - \mu(\pm I_\xi)) \\ &= 2\mu(I_x) - m \\ &\geq (3/2)\mu(I_x), \end{aligned}$$

which proves the corollary. □

Proof of Theorem 1. Consider the orbit of the critical value $\{c = c_0, c_1, c_2, \ldots\}$, where $c_n = F^{\circ n}(c)$. Let $m = \mu(I_c) > 0$, and apply Lemma 10 to the point $x = c$. Clearly the only possible cases are (a) and (c), since $\pi(c) \notin (-\pi(\xi), \pi(\xi))$.

In case (c) we obtain the estimate $\mu(I_{c_1}) \geq 2m$. This, by repeated application of Corollary 3, will lead to the estimate $\mu(I_{c_{n+1}}) \geq (3/2)^n \mu(I_{c_1})$ which tends to infinity as $n \to \infty$ and therefore is impossible.

In case (a), I_c and $-I_\xi$ overlap along some I_y with $F(y) \in (-\beta, \pi(c))$ and $\mu(I_{c_1}) = \mu[c_1, F(y)] = 2\mu[c, y] \geq m$. Apply Lemma 10 this time to $x = c_1$. Note that the only possible case is (c), since $\pi(c_1) \in [-\beta, \pi(c))$. This gives the estimate $\mu(I_{c_2}) = 2\mu(I_{c_1}) \geq 2m$. Hence successive applications of Corollary 3 will give the estimate $\mu(I_{c_{n+1}}) \geq (3/2)^n \mu(I_{c_2})$, which again contradicts the fact that the Brolin measure of the Julia set is finite.

The contradiction shows that $m = \mu(I_c)$ must be zero, and this completes the proof of Theorem 1 by Lemma 9. □

1.5. **Further Discussion.** Finally, we consider the following result, which is a consequence of Theorem 1 as well as the fact that the Julia set has no compact forward-invariant proper subsets of positive Brolin measure.

Theorem 2. *Let $f : z \mapsto z^2 + c$ be a quadratic polynomial with locally-connected filled Julia set K. If we exclude the Chebyshev case and the cases where the α-fixed point of f is attracting or $\alpha = \beta$, then every embedded arc in K has Brolin measure zero.*

The exceptional cases correspond respectively to $c = -2$ where the Julia set is a straight line segment, c in the "main cardioid" of the Mandelbrot set where the Julia set is a quasi-circle, and $c = 1/4$ where the Julia set is a Jordan curve but not a quasi-circle. Roughly speaking, the theorem says that in any other case, embedded arcs are buried in the filled Julia set so that they are almost invisible from the basin of infinity.

We make the following elementary observation for the proof:

Lemma 11. *Let $A \subset J$ be forward-invariant under f, i.e., $f(A) \subset A$. Then either $\mu(A) = 0$ or $\mu(A) = 1$. In particular, if A is compact and $A \neq J$, then $\mu(A) = 0$.*

Proof. Let $\gamma : \mathbb{T} \to J$ be the Carathéodory loop and $E = \gamma^{-1}(A)$. Then E is forward-invariant under the doubling map $d : \mathbb{T} \to \mathbb{T}$ defined by $d(t) = 2t \pmod 1$. We prove that $\ell(E) = 0$ or $\ell(E) = 1$, where ℓ denotes the Lebesgue measure on $\mathbb{T}$. Let $\ell(E) > 0$ and let x be a point of density of E. Given an $\varepsilon > 0$, we can find an $n > 0$ and an interval $S \subset \mathbb{T}$ centered at x such that $\ell(S) = 2^{-n}$ and $\ell(S \cap E) \geq (1-\varepsilon)\ell(S)$. Apply the n-th iterate $d^{\circ n}$ on S and use $d^{\circ n}(E) \subset E$ to estimate

$$1 - \varepsilon \leq 2^n \ell(S \cap E) = \ell(d^{\circ n}(S \cap E)) \leq \ell(\mathbb{T} \cap E) = \ell(E).$$

Since this is true for every $\varepsilon > 0$, we must have $\ell(E) = 1$. □

Corollary 4. *Still assuming that K is locally-connected, the Brolin measure of the union of the boundaries of bounded Fatou components of f is zero unless the α-fixed point is attracting or $\alpha = \beta$ in which case the corresponding measure is one.*

Proof. Since every bounded Fatou component eventually enters a cycle of Fatou components of the form $U_1 \mapsto U_2 \mapsto \ldots \mapsto U_p \mapsto U_1$, it suffices to prove that $\mu(A) = 0$, where $A = \bigcup_{j=1}^{p} \partial U_j$. This set is compact and forward-invariant under f, so by Lemma 11 if $\mu(A) > 0$, then $A = J$ must be the case. But this implies that f has only p bounded Fatou components. It is easy to see that this can happen only if $p = 1$, in which case the component is either the immediate basin of attraction for an attracting fixed point or the attracting petal for a parabolic fixed point. $\square$

As an illustrative example, consider a quadratic polynomial f whose α-fixed point is the center of a Siegel disk U with rotation number θ of constant type (an example is provided by $f : z \mapsto z^2 - 0.3905408 - 0.5867879i$, where $\theta = (\sqrt{5} - 1)/2$ is the golden mean). By [**Pe**], the filled Julia set is locally-connected. The critical point $0 \in \partial U$ is the landing point of exactly two rays $(R_s, R_{s+1/2})$, where

$$s = \sum_{0<p/q<\theta} 2^{-(q+1)}. \tag{1.8}$$

Since the orbit of 0 is dense on ∂U, the set of angles t for which $\gamma(t) \in \partial U$ coincides with the closure of the orbit of s under the doubling map on the circle. This set is known to be an invariant Cantor set C of measure zero in the interval $[s, s + 1/2] \subset \mathbb{T}$ [**B-S**]. It follows that the set of all t for which $\gamma(t)$ belongs to the boundary of a bounded Fatou component is the countable union of Cantor sets consisting of C and all its preimages under the doubling map. This set has Lebesgue measure zero, hence the union of the boundaries of all bounded Fatou components will have Brolin measure zero.

Proof of Theorem 2. Let $\eta \subset K$ be any embedded arc. Let B be the set of biaccessible points in J and B' be the set of all points in J which belong to the boundary of a bounded Fatou component. By Theorem 1 and Corollary 4, we have $\mu(B) = \mu(B') = 0$. On the other hand, by Lemma 3(b), every $z \in \eta \cap J$ is either an endpoint or it belongs to $B \cup B'$. Hence, $\mu(\eta) = \mu(\eta \cap J) \leq \mu(B \cup B') = 0$. $\square$

Corollary 5. *A locally-connected quadratic Julia set is not a countable union of embedded arcs unless it is a straight line or a Jordan curve.*

Part II: The Siegel and Cremer Cases

2.1. **Introduction.** Consider a quadratic polynomial

$$f : z \mapsto z^2 + c \tag{2.1}$$

in the complex plane $\mathbb{C}$. A fixed point $z = f(z)$ is called *indifferent* if the *multiplier* $\lambda = f'(z)$ has the form $e^{2\pi i\theta}$, where the *rotation number* θ belongs to $\mathbb{R}/\mathbb{Z}$. We call z *irrationally indifferent* if θ is irrational so that λ is on the unit circle but not a root of unity.

Let z be an irrationally indifferent fixed point of f. When f is holomorphically linearizable about z, we call z a *Siegel* fixed point. On the other hand, when z is non-linearizable, it is called a *Cremer* fixed point.

The two fixed points α and β have multipliers $\lambda = 2\alpha$ and $2 - \lambda = 2\beta$. It follows that only the α-fixed point can be indifferent. The critical value parameter c is then given by

$$c = \lambda(2 - \lambda)/4.$$

Therefore, the set of all quadratic polynomials which have an indifferent fixed point is a cardioid in the c-plane parametrized by λ on the unit circle. The set of quadratic polynomials with an irrationally indifferent fixed point is then a dense subset of this cardioid. We call a quadratic polynomial f in (2.1) *Siegel* or *Cremer* if the α-fixed point is irrationally indifferent and has the corresponding property.

It follows from classical Fatou-Julia theory that the filled Julia set K and the Julia set $J = \partial K$ are connected when f is Siegel or Cremer. Every connected component of the interior of K is a topological disk called a *bounded Fatou component* of f. In the Siegel case, the component S of the interior of K containing the fixed point α is called the *Siegel disk* of f on which the action of f is holomorphically conjugate to the rigid rotation $z \mapsto e^{2\pi i\theta} z$.

Since $f(z) = f(-z)$ by (2.1), J is invariant under the 180° rotation $\tau : z \mapsto -z$. If U is an open Jordan domain in the plane such that $\overline{U} \cap \tau(\overline{U}) = \emptyset$, it follows that f is univalent in some Jordan domain V containing the closure $\overline{U}$.

According to Fatou and Sullivan, every bounded Fatou component must eventually map to the immediate basin of attraction of an attracting periodic point, or to an attracting petal for a parabolic periodic point, or to a periodic Siegel disk for f [**Mi1**]. On the other hand, by [**Do1**] a polynomial of degree $d \geq 2$ can have at most $d - 1$ non-repelling periodic orbits. It follows that in the Siegel case, every bounded Fatou component eventually maps to the Siegel disk S centered at α. In the Cremer case, however, we simply conclude that K has no interior, so that $K = J$.

2.2. **Arithmetical conditions.** It is well-known that the behavior of orbits near the indifferent fixed point is intimately connected to the arithmetical properties of the rotation number $0 < \theta < 1$. There are certain classes of irrational numbers which are of special interest in holomorphic dynamics and in this paper we will be working with some of them. Let

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

be the continued fraction expansion of θ, where all the a_i are positive integers, and

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

be the n-th rational approximation of θ. We say that

- θ is of *constant type* (we write $\theta \in \mathcal{CT}$) if $\sup_n a_n < +\infty$.
- θ is *Diophantine* (we write $\theta \in \mathcal{D}$) if there exist positive constants C and ν such that for every rational number $0 \leq p/q < 1$, we have $|\theta - p/q| > C/q^\nu$. This condition is equivalent to $\sup_n(\log q_{n+1}/\log q_n) < +\infty$.
- θ is of *Yoccoz type* (we write $\theta \in \mathcal{H}$) if every analytic circle diffeomorphism with rotation number θ is analytically linearizable. (An explicit arithmetical description of $\mathcal{H}$ is given by Yoccoz although it is not easy to explain; see [**Yo2**].)

  A closely related condition, which we denote by $\mathcal{H}'$, is defined as follows: $\theta \in \mathcal{H}'$ if and only if every analytic circle diffeomorphism with rotation number θ, with no periodic orbit in some neighborhood of the circle, is analytically linearizable [**PM1**].
- θ is of *Brjuno type* (we write $\theta \in \mathcal{B}$) if it satisfies the condition

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty, \tag{2.2}$$

We have the proper inclusions $\mathcal{H} \subset \mathcal{H}'$ and $\mathcal{CT} \subset \mathcal{D} \subset \mathcal{H} \subset \mathcal{B}$. It is not hard to show that $\mathcal{D}$, hence $\mathcal{H}, \mathcal{H}'$ and $\mathcal{B}$, are sets of full measure in $\mathbb{R}/\mathbb{Z}$.

By the theorem of Brjuno-Yoccoz [**Yo1**], f is a Siegel quadratic polynomial if and only if $\theta \in \mathcal{B}$.

2.3. **Basic results.** Very little is known about the topology of the Julia set of f in the Siegel or Cremer case or the dynamics of f on its Julia set. The following theorem summarizes the basic results in the Cremer case:

Theorem 3. *Let f in* (2.1) *be a Cremer quadratic polynomial, so that $\theta \notin \mathcal{B}$. Then*

(a) *The Julia set J cannot be locally-connected* [**Su**].
(b) *Every neighborhood of the Cremer fixed point α contains infinitely many repelling periodic orbits of f* [**Yo1**].
(c) *The critical point* 0 *is recurrent, i.e., it belongs to the closure of its orbit* $\{f^{\circ n}(0)\}_{n>0}$ [**Ma**].
(d) *The critical point* 0 *is not accessible from* $\mathbb{C} \smallsetminus J$ [**Ki**].

See also [**Sø**] for the so-called "Douady's non-landing theorem," which says that for a generic Cremer quadratic polynomial there is an external ray which accumulates on the Cremer fixed point and its preimage. Perez-Marco has shown that for every Cremer quadratic polynomial there exists an external ray whose prime-end impression contains the Cremer fixed point and its preimage [**PM2**]. Both results shed some light on why the Julia set of a Cremer quadratic polynomial fails to be locally-connected.

In the Siegel case, we know a little bit more, but still the situation is far from being fully understood.

Theorem 4. *Let f in* (2.1) *be a Siegel quadratic polynomial, so that $\theta \in \mathcal{B}$. Let S denote the Siegel disk of f. Then*

- (a) *If $\theta \in \mathcal{CT}$, then the boundary ∂S is a quasi-circle which contains the critical point* 0 [**Do2**]. *The Julia set J is locally-connected and has measure zero* [**Pe**].
- (b) *If $\theta \in \mathcal{H}$, then $0 \in \partial S$* [**He1**].
- (c) *For some rotation numbers $\theta \in \mathcal{B} \smallsetminus \mathcal{H}$, the entire orbit of* 0 *is disjoint from ∂S* [**He2**]. *In this case, J cannot be locally-connected* [**Do2**].
- (d) *For every $\theta \in \mathcal{B}$, the critical point* 0 *is recurrent.*

Part (b) was proved by Herman for $\theta \in \mathcal{D}$, but his proof works equally well for $\theta \in \mathcal{H}$. We will include a very short proof for the latter case in section 2.4. The proof of part (d) goes as follows: By classical Fatou-Julia theory, every point in ∂S is in the closure of the orbit of 0 [**Mi1**], so recurrence is immediate if $0 \in \partial S$. If $0 \notin \partial S$ and 0 is not recurrent, then by [**Ma**] the invariant set ∂S is expanding, i.e., there is a constant $\lambda > 1$ and a positive integer k such that $|(f^{\circ k})'(z)| > \lambda$ for all $z \in \partial S$. It follows that the same inequality holds over some neighborhood U of ∂S, and we may as well assume that $U \cap S$ is invariant. Take a small disk $V \Subset U \cap S$. Since $f^{\circ k}|_{U \cap S}$ is holomorphically conjugate to the rigid rotation $z \mapsto e^{2\pi i k \theta} z$, there exists a sequence $n_j \to \infty$ such that $f^{\circ k n_j}$ converges uniformly to the identity map on V as $j \to \infty$. But this is impossible since for all $z \in V$, $|(f^{\circ k n_j})'(z)| > \lambda^{n_j} \to \infty$.

Comparing the two theorems, we notice that the Cremer case and the Siegel case with $0 \notin \partial S$ share many properties. This is a general philosophy which is partially explained by the theory of "hedgehogs" introduced recently by Perez-Marco [**PM1**] (see section 2.4 below).

Inspired by this similarity, one expects the following to be true:

Conjecture. *Let f be a Siegel quadratic polynomial and $0 \notin \partial S$. Then*

- (i) *Every neighborhood of ∂S contains infinitely many repelling periodic orbits of f.*
- (ii) *The critical point* 0 *is not accessible from $\mathbb{C} \smallsetminus K$.*

By an argument similar to [**Ki**], one can show that (i) implies (ii) (see also Proposition 3).

2.4. **Hedgehogs.** Let f be a Siegel or Cremer quadratic polynomial as in (2.1). Let U be a simply connected domain with compact closure which contains the closure of the Siegel disk S in the linearizable case, or the indifferent fixed point α in the non-linearizable case. Suppose that f is univalent in a neighborhood of the closure $\overline{U}$. Then there exists a set $H = H_U$ with the following properties:

(i) $\alpha \in H \subset \overline{U}$,
(ii) H is compact, connected and full,
(iii) $\partial H \cap \partial U$ is nonempty,
(iv) $\partial H \subset J$,
(v) $f(H) = H$.

Note that H has nonempty interior if and only if α is linearizable. In this case our assumption that f is univalent on U implies that the critical point is off the boundary of the Siegel disk. Clearly $H \supset \overline{S}$.

Such an H is called a *hedgehog* for the restriction $f|_U : U \to \mathbb{C}$. See Fig. 10(a) for the Cremer case and (b) for the Siegel case. (We would like to emphasize that the topology of a hedgehog is infinitely more complicated than anything we can possibly sketch!) The existence of such totally invariant sets is proved by Perez-Marco [**PM1**].

Note that in the Siegel case, one can get totally invariant sets H with the above properties (i)-(v) even if ∂U intersects the closure $\overline{S}$. But in this case the existence of H is not hard to show because we can simply take H as $\overline{S}$ or a compact invariant piece with analytic boundary inside the Siegel disk (see Fig. 10(c) and (d)).

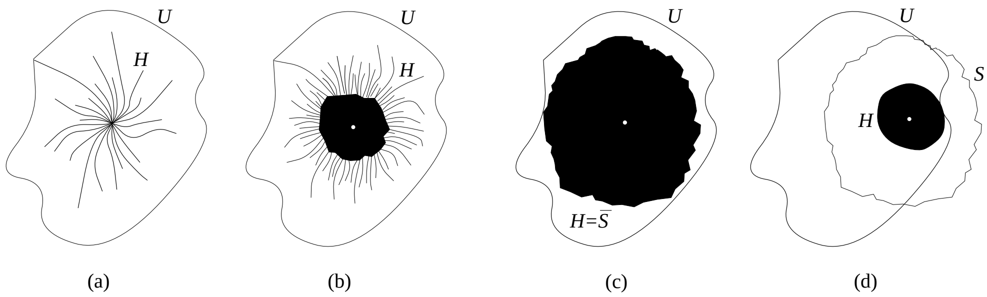

FIGURE 10

Hedgehogs turn out to be useful because of the following nice construction: Uniformize the complement $\mathbb{C} \smallsetminus H$ by the Riemann map $\phi : \mathbb{C} \smallsetminus H \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ and consider the induced map $g = \phi \circ f \circ \phi^{-1}$ which is defined (by (v) above) and holomorphic in an open annulus $\{z \in \mathbb{C} : 1 < |z| < r\}$. Use the Schwarz Reflection Principle to extend g to the annulus $\{z \in \mathbb{C} : r^{-1} < |z| < r\}$. The restriction of g to the unit circle $\mathbb{T}$ will then be a real-analytic diffeomorphism whose rotation number is exactly

$\frac{1}{2\pi i}\log f'(\alpha) = \theta \in \mathbb{R}/\mathbb{Z}$ (see [**PM1**]). This allows us to transfer results from the more developed theory of circle diffeomorphisms to the less explored theory of indifferent fixed points of holomorphic maps.

Using the above construction, it is not hard to prove the following fact (see [**PM2**]):

Proposition 1. *Let p be a point in a hedgehog H which is biaccessible from outside of H. Then $p \in \partial S$ in the Siegel case and $p = \alpha$ in the Cremer case.*

In fact, let us assume that we are in the Siegel case and $p \notin \partial S$. Then one can find a simple arc γ in $\mathbb{C} \smallsetminus H$ which starts and terminates at p and does not encircle the indifferent fixed point α. Let D be the bounded connected component of $\mathbb{C} \smallsetminus (H \cup \gamma)$. Evidently $\overline{D}$ is disjoint from $\overline{S}$. The topological disk $D' = \phi(D)$ is bounded by the simple arc $\phi(\gamma)$ and an interval I on the unit circle. (The fact that $\phi(\gamma)$ actually lands from both sides on the unit circle follows from general theory of conformal mappings; see for example [**Po**], Proposition 2.14.) Since g has irrational rotation number on the unit circle $\mathbb{T}$, for some integer N we have $\bigcup_{i=0}^{N} g^{\circ i}(I) = \mathbb{T}$. By choosing γ close enough to H, we can assume that $g, g^{\circ 2}, \cdots, g^{\circ N}$ are all defined on D' and $\bigcup_{i=0}^{N} g^{\circ i}(D')$ contains an entire outer neighborhood of $\mathbb{T}$. It follows that $\bigcup_{i=0}^{N} f^{\circ i}(D)$ covers an entire deleted neighborhood of H. Therefore, some iterate $f^{\circ i}(\overline{D})$ intersects ∂S. Since $f^{\circ i}$ is univalent on $\overline{D} \cup \overline{S}$, it follows that $\overline{D} \cap \partial S \neq \emptyset$, which contradicts our assumption. The proof in the Cremer case is similar.

The construction of the circle maps associated with hedgehogs as described above gives short proofs for some interesting facts. As the first example, we prove that there are no periodic points on ∂S when the critical point 0 is off this boundary, a fact that will be used in the proof of Theorem 5. One can find a proof of this result in [**PM1**] for indifferent germs, but the fact that we are working with polynomials makes the proof even shorter.

First we need the following lemma:

Lemma 12. *Let f be a Siegel quadratic polynomial as in* (2.1) *whose critical point* 0 *is off the boundary ∂S of the Siegel disk. Then the closure $\overline{S}$ is full and f acts injectively on it.*

It is reasonable to speculate that the closure of any bounded Fatou component for a quadratic polynomial is full. This is known to be true except when the polynomial has a periodic Siegel disk S with the critical point on its boundary ∂S. In this case, we do not know if ∂S can separate the plane into more than two connected components (a so-called "Lakes of Wada" example in plane topology [**H-Y**]).

Proof. (Compare [**He1**], [**PM2**]) Since $f(z) = f(-z)$ for all z, if f is not injective on $\overline{S}$, there must be a pair of symmetric points p and $-p = \tau(p)$ in ∂S. Since J has a $180°$ rotational symmetry, $f^{-1}(S) = S \cup \tau(S)$. So p and $-p$ also belong to $\partial(\tau(S))$.

Consider the connected component V of $\mathbb{C} \smallsetminus (\overline{S} \cup \overline{\tau(S)})$ which contains the critical point 0. Since V is open and $\partial V \subset J$, it follows from the Maximum Principle that V has to be a bounded Fatou component of f. This contradicts the fact that $0 \in J$.

Let us now assume that $\overline{S}$ is not full and let U be a bounded component of $\mathbb{C} \smallsetminus \overline{S}$. Since $\partial U \subset \partial S \subset J$, it follows again from the Maximum Principle that U has to be a bounded Fatou component of f, hence it eventually maps to S, i.e., $f^{\circ n}(U) = S$ for some $n \geq 1$. Therefore $f^{\circ n-1}(U) = \tau(S)$. But the boundary of $f^{\circ n-1}(U)$ is a subset of ∂S, which implies that the common boundary $\partial S \cap \partial(\tau(S))$ is nonempty. This contradicts the fact that $f|_{\partial S}$ is injective. $\square$

Proposition 2. *Let f be a Siegel quadratic polynomial whose critical point 0 is off the boundary ∂S. Then there are no periodic points on ∂S.*

Proof. By the above lemma $\overline{S}$ is full and f acts injectively on it, so we can find a Jordan domain U containing $\overline{S}$ such that $f|_{\overline{U}}$ is univalent. Consider a hedgehog $H = H_U$ for the restriction $f|_U$. Clearly $H \supset \overline{S}$. Suppose that there is a periodic point on ∂S which is necessarily repelling. Then there exists a rational external ray R landing at this point, hence $f^{\circ n}(R) = R$ for some $n \geq 1$ (see for example [**Mi1**]). Consider the induced map $g = \phi \circ f \circ \phi^{-1}$ as described above, and look at the arc $\gamma = \phi(R)$. It is a standard fact that γ has to land at some point $p \in \mathbb{T}$ [**Po**] and $g^{\circ n}(p) = p$. But this contradicts the fact that the rotation number of g is irrational. $\square$

In the second application, we prove Theorem 4(b): We want to show that $\theta \in \mathcal{H}$ implies $0 \in \partial S$. If not, by Lemma 12 $\overline{S}$ is full and $f|_{\overline{S}}$ is univalent. Consider a Jordan domain U, a hedgehog H_U and the induced circle map g as in the above proof. Since the rotation number of g belongs to $\mathcal{H}$, g is analytically linearizable. The linearization can be extended holomorphically to an annulus neighborhood of the unit circle $\mathbb{T}$. Pulling this neighborhood back, we find a larger domain containing S on which f is linearizable, which contradicts the definition of a Siegel disk.

As a final application, we prove the following:

Proposition 3. *Let f be a Siegel quadratic polynomial whose critical point 0 is off the boundary ∂S. If $\theta \in \mathcal{H}'$, the critical point 0 is not accessible from $\mathbb{C} \smallsetminus K$.*

Proof. Consider the hedgehog construction as in the proof of Proposition 2 or the above proof for Theorem 4(b). If there are no periodic orbits in some neighborhood of ∂S, it follows that g has no periodic orbit in some neighborhood of $\mathbb{T}$ either. Since the rotation number of g is $\theta \in \mathcal{H}'$, g has to be linearizable. Now we can get a contradiction as in the above proof for Theorem 4(b). So every neighborhood of ∂S must contain infinitely many periodic orbits. The fact that this implies non-accessibility of 0 follows easily by an argument similar to [**Ki**]. $\square$

2.5. **Wakes.** To see the behavior of rays near infinity, it will be convenient to add a circle at infinity $\mathbb{T}_\infty \simeq \mathbb{R}/\mathbb{Z}$ to the complex plane to obtain a closed disk ⓒ topologized in the natural way. We denote the point $\lim_{r\to\infty} re^{2\pi it}$ on $\mathbb{T}_\infty$ simply by $\infty \cdot e^{2\pi it}$. The action of f in (2.1) on the complex plane extends continuously to ⓒ by

$$f(\infty \cdot e^{2\pi it}) = \infty \cdot e^{4\pi it}, \tag{2.3}$$

which is just the doubling map on $\mathbb{T}_\infty$. Note that the symmetry $f(z) = f(-z)$ also extends to ⓒ if we define $-\infty \cdot e^{2\pi it} = \infty \cdot e^{2\pi i(t+1/2)}$.

Definition. Let f be a quadratic polynomial as in (2.1) with connected Julia set. Let $z \neq \alpha$ be a biaccessible point in J with two distinct rays R and R' landing on it. We call (R, R') a *ray pair.* By the Jordan Curve Theorem, $R \cup R' \cup \{z\}$ cuts the plane into two open topological disks. By the *wake* W of the ray pair (R, R') we mean the connected component of $\mathbb{C} \smallsetminus (R \cup R' \cup \{z\})$ which does not contain the fixed point α. The other component is called the *co-wake* and it is denoted by $\check{W}$. Point z is called the *root* of W. The *angle* $a(W)$ of the wake is just the (normalized) measure of $\overline{W} \cap \mathbb{T}_\infty$. Clearly $a(W) + a(\check{W}) = 1$ (see Fig. 11(a)).

Since distinct external rays are disjoint, it follows that any two wakes with distinct roots are either disjoint or nested.

In the following lemma we collect basic properties of wakes (compare [**G-M**] or [**Mi2**]):

Lemma 13. *Let $z \in J$ be a biaccessible point, $z \neq \alpha$, and let W be a wake with root z.*

(a) *If $z \neq 0$, then $a(W) > 1/2$ if and only if W contains the critical point 0.*
(b) *If $a(W) = 1/2$, then z must be the critical point 0. Conversely, if there is any ray R landing at 0, then $R' = \tau(R)$ also lands at 0 and the two rays span a wake W with $a(W) = 1/2$.*
(c) *Let $a(W) < 1/2$ and $f(z) \neq \alpha$. Then $f(W)$ is a wake or a co-wake with root $f(z)$, depending on whether $-\alpha \notin W$ or $-\alpha \in W$. In either case, $f : W \to f(W)$ is a conformal isomorphism and $a(f(W)) = 2a(W)$.*

Proof. Let W be the wake of a ray pair (R, R').

(a) Let $0 \in W$ and $a(W) < 1/2$. Consider the symmetric region $\tau(W)$ whose angle is equal to $a(W)$. W and $\tau(W)$ intersect since both contain 0 (see Fig. 11(b)). On the other hand, $\overline{W} \cap \overline{\tau(W)} \cap \mathbb{T}_\infty = \emptyset$ because $a(W) < 1/2$. Since $\overline{W}$ and $\overline{\tau(W)}$ are both homeomorphic to closed disks, it follows that the ray pairs (R, R') and $(\tau(R), \tau(R'))$ must intersect, which is a contradiction. Therefore $a(W) > 1/2$ if $0 \in W$.

On the other hand, let $a(W) > 1/2$. Then the angle of the co-wake $\check{W}$ has to be less than $1/2$, so by the above argument $0 \notin \check{W}$, or $0 \in W$. This proves (a).

(b) If $a(W) = 1/2$, then $R' = \tau(R)$. Hence $z = \tau(z)$ by continuity, which means $z = 0$. The converse is trivial.

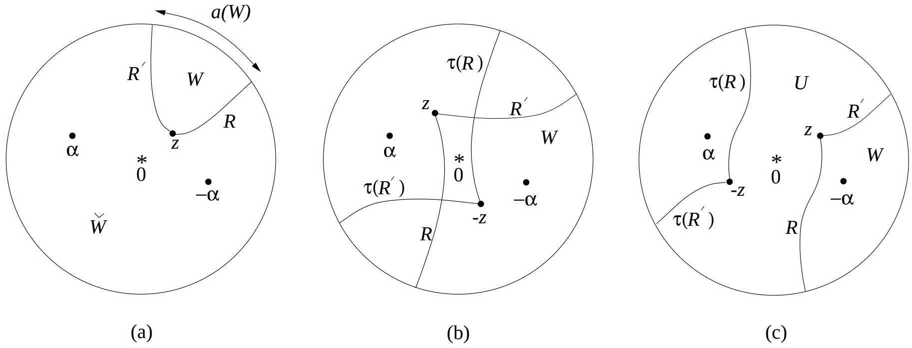

FIGURE 11

(c) If $a(W) < 1/2$, then the ray pairs (R, R') and $(\tau(R), \tau(R'))$ cut the plane into simply connected domains W, $\tau(W)$ and an open set U which is either a simply connected domain or the disjoint union of two simply connected domains depending on whether $z \neq 0$ or $z = 0$. By (a), $0 \notin W \cup \tau(W)$. Consider the ray pair $(f(R), f(R'))$ landing at $f(z)$, and let W' be the corresponding wake. The pull-back of W' by f either consists of the disjoint union $W \sqcup \tau(W)$ or the open set U (see Fig. 11(c)). In the first case, f maps W to W' isomorphically and $-\alpha \notin W$. In the second case, however, we must have $-\alpha \in W$, $\alpha \in \tau(W)$, and both W and $\tau(W)$ map isomorphically to the co-wake $\check{W}'$. The fact that $a(f(W)) = 2a(W)$ simply follows from (2.3). $\square$

2.6. **The main theorem.** Now we are in a position to state and prove the main theorem of Part II:

Theorem 5. *Let f be a quadratic polynomial as in* (2.1) *which has an irrationally indifferent fixed point α. Let z be a biaccessible point in the Julia set of f. Then:*

- *In the Siegel case, the orbit of z must eventually hit the critical point* 0.
- *In the Cremer case, the orbit of z must eventually hit the fixed point α.*

(Compare [**S-Z**] where this same result for the Cremer case is proved by a somewhat different argument.)

In the Siegel case, if the critical point 0 is accessible, then exactly two rays land there (see the proof of Lemma 13(b)). This happens, for example, when $\theta \in \mathcal{CT}$, since in this case by Theorem 4(a) the Julia set is locally-connected. On the other hand, for some rotation numbers $\theta \in \mathcal{B} \cap \mathcal{H}'$, the critical point is not accessible so that there are no biaccessible points in the Julia set (see Corollary 6).

In the Cremer case, if the fixed point α is accessible, then infinitely many rays land there. In fact, if R_t lands at α, then t is irrational and every $R_{2^n t}$ lands at α also. However, there is no known example where one can decide whether α is accessible or not.

The proof of Theorem 5 is based on the following lemma:

Lemma 14. *Let f be a Siegel or Cremer quadratic polynomial as in* (2.1)*. Assume that there exists a biaccessible point in J whose orbit never hits the critical point* 0 *or the fixed point α. Then there exists a ray pair which separates α from* 0*.*

Proof. Let $z \in J$ be such a biaccessible point and (R, R') be a ray pair which lands at z. Consider the associated wake W_0 with root z. Since $z \neq 0$, we have $a(W_0) \neq 1/2$ by Lemma 13(b). If $a(W_0) > 1/2$, then $0 \in W_0$ by Lemma 13(a) and (R, R') separates α from 0. Let us consider the case where $a(W_0) < 1/2$. If $-\alpha \in W_0$, then (R, R') must separate $-\alpha$ from 0 because by Lemma 13(a), $0 \notin W_0$. It follows that the symmetric ray pair $(\tau(R), \tau(R'))$ separates α from 0. If, however, $-\alpha \notin W_0$, then by Lemma 13(c), $W_1 = f(W_0)$ is a wake with root $z_1 = f(z)$ with angle $a(W_1) = 2a(W_0)$.

Now we can replace W_0 by W_1 in the above argument. If either $a(W_1) > 1/2$ or $a(W_1) < 1/2$ and $-\alpha \in W_1$, we can find a ray pair separating α from 0. Otherwise, we consider the new wake $W_2 = f(W_1)$ with angle $a(W_2) = 2^2 a(W_0)$. Since each passage $W_i \mapsto W_{i+1}$ implies doubling the angles, this process must stop at some stage, and this proves the lemma. $\square$

Proof of Theorem 5. It will be more convenient to consider the Cremer case first. Suppose that the orbit of z never hits α. Since the critical point is not accessible by Theorem 3(d), Lemma 14 gives us a ray pair (R, R') landing at some point $p \in J$ which separates α from 0. Let W be the corresponding wake with root p and consider the co-wake $\check{W}$. The restriction of f to the closure of $\check{W}$ is univalent since otherwise this closure would intersect the closure of the symmetric domain $\tau(\check{W})$, which is impossible since $a(\check{W}) < 1/2$. To work with a Jordan domain in the plane we cut off $\check{W}$ along an equipotential curve and call the resulting domain U (see Fig. 12(a)).

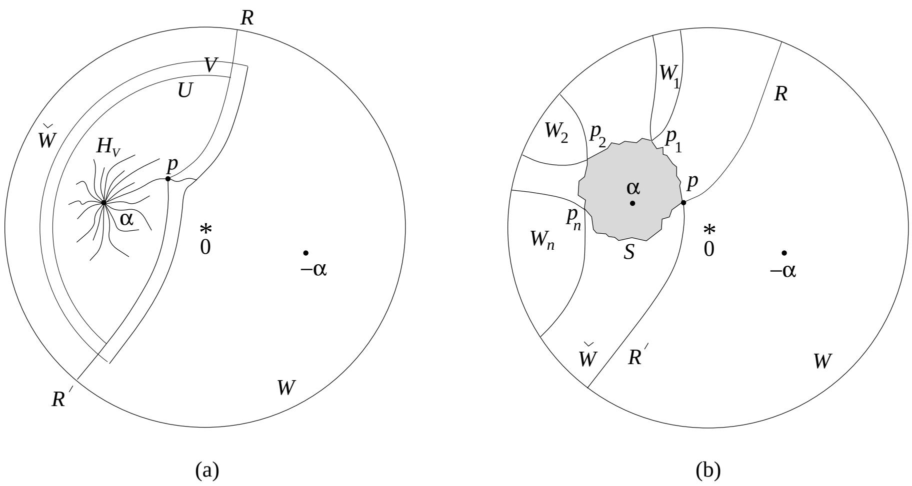

FIGURE 12

Let us consider a slightly larger Jordan domain $V \supset \overline{U}$ with compact closure such that $f|_{\overline{V}}$ is still univalent. The hedgehog H_V for the restriction $f|_V : V \to \mathbb{C}$ has to reach the boundary of V. Since H_V is connected and intersects U, it has to intersect the boundary of U as well. But $H_V \subset J$ and $\partial U \cap J = \{p\}$. Hence $p \in H_V$. Since p is biaccessible from outside of the Julia set, it follows that $H_V \smallsetminus \{p\}$ is disconnected. Therefore, p is biaccessible from outside of H_V. This contradicts Proposition 1, and finishes the proof of the theorem in the Cremer case.

Let us now assume that we are in the Siegel case. If the orbit of z eventually hits the critical point 0, there is nothing to prove. Otherwise, since this orbit trivially cannot hit the fixed point $\alpha \in S$, we are again in the situation of Lemma 14. Therefore, there exists a ray pair (R, R') landing at a point $p \in J$ which separates α from 0. In particular the critical point 0 is off the boundary ∂S of the Siegel disk. Then the same argument as in the Cremer case with an application of Proposition 1 shows that p must belong to ∂S.

As before, let W be the wake of the ray pair (R, R'), with root p. Then by construction W contains the critical point 0 while the co-wake $\check{W}$ contains the Siegel disk S and has its boundary touching $\overline{S}$ only at p. The point p is not periodic by Proposition 2. Hence the successive images $p_n = f^{\circ n}(p) \in \partial S$ are all contained in $\check{W}$ for $n \geq 1$. Therefore each wake W_n corresponding to the ray pair $(f^{\circ n}(R), f^{\circ n}(R'))$, with root point p_n, is also contained in $\check{W}$ (see Fig. 12(b)). In particular, none of these wakes contains the critical point. Hence $a(W_{n+1}) = 2a(W_n) < 1/2$ for all n by Lemma 13(c), which is clearly impossible. The contradiction shows that the orbit of z must eventually hit the critical point. □

By Proposition 3, we have the following corollary:

Corollary 6. *Let f be a Siegel quadratic polynomial with $0 \notin \partial S$ and $\theta \in \mathcal{H}'$. Then there are no biaccessible points in J at all.*

By Lemma 13(b), every wake with angle 1/2 must have its root at the critical point 0. The converse is not true for arbitrary quadratic polynomials. For example, the real Feigenbaum map $z \mapsto z^2 - 1.401155\cdots$ has four distinct external rays landing on its critical point (compare with [**J-H**]). However, in the case of a Siegel quadratic polynomial, the critical point 0 is the landing point of *at most* one ray pair $(R_s, \tau(R_s))$ (In the Cremer case, there are no such ray pairs by Theorem 3(d)). This is nontrivial and follows from the statement that every Siegel or Cremer quadratic on the boundary of the main cardioid of the Mandelbrot set is the landing point of a unique parameter ray [**G-M**]. In fact, one can explicitly compute the angle s of the candidate ray pair $(R_s, \tau(R_s))$ which may land at 0 from the equation (1.8) in Part I. It is interesting that the uniqueness of such s also follows from Theorem 5:

Corollary 7. *Let f be a Siegel quadratic polynomial as in* (2.1). *Then, no point in the Julia set J is the landing point of more than two rays. In particular, at most one ray pair lands at the critical point* 0.

Proof. By Theorem 5 it suffices to prove the corollary for the critical point. Suppose that there is a ray pair (R, R') which lands at 0 such that $R' \neq \tau(R)$. It follows that $(f(R), f(R'))$ is a ray pair which lands at the critical value c. By Theorem 5, the orbit of c must eventually hit the critical point 0. But this means that 0 is periodic, which is impossible. $\square$

References

[**Br**] H. Brolin, *Invariant sets under iteration of rational functions*, Arkiv för Math., **6** (1965) 103-144.

[**B-S**] S. Bullett and P. Sentenac, *Ordered orbits of the shift, square roots, and the devil's staircase*, Math. Proc. Camb. Phil. Soc., **115** (1994) 451-481.

[**Do1**] A. Douady, *Systemes dynamiques holomorphes*, Asterisque, **105-106** (1983) 36-63.

[**Do2**] A. Douady, *Disques de Siegel at aneaux de Herman*, Seminar Bourbaki, Asterisque, **152-153** (1987) 151-172.

[**Do3**] A. Douady, *Description of compact sets in* $\mathbb{C}$, in Topological Methods in Modern Mathematics, A symposium in honor of John Milnor's sixtieth birthday, Publish or Perish, 1993, 429-466.

[**D-H**] A. Douady and J. Hubbard, *Etude dynamique des polynomes complexes I, II*, Orsay Notes, 1984-85.

[**E-L**] A. Eremenko and M. Lyubich, *The dynamics of analytic transformations*, Leningrad Math. J., **1** (1990) 563-634.

[**G-M**] L. Goldberg and J. Milnor, *Fixed points of polynomial maps II*, Ann. Sci. Ec. Norm. Sup., **26** (1993) 51-98.

[**He1**] M. Herman, *Are there critical points on the boundaries of singular domains?*, Comm. Math. Phys., **99** (1985) 593-612.

[**He2**] M. Herman, *Conjugaison quasisymetrique des diffeomorphismes des cercle a des rotations et applications aux disques singuliers de Siegel*, Manuscript.

[**H-Y**] J. Hocking and G. Young, *Topology*, Addison Wesley, 1961.

[**J-H**] Y. Jiang and J. Hu, *The Julia set of the Feigenbaum quadratic polynomial is locally-connected*, Preprint, 1993.

[**Ki**] J. Kiwi, *Non-accessible critical points of Cremer polynomials*, SUNY at Stony Brook IMS preprint, 1996/2.

[**Ly**] M. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergod. Th. and Dyam. Sys., **3** (1983) 351-385.

[**Ma**] R. Mañe, *On a lemma of Fatou*, Bol. Soc. Bras. Mat., **24** (1993) 1-12.

[**Mc**] C. McMullen, *Complex Dynamics and Renormalization*, Annals of Math Studies, vol. 135, 1994.

[**Mi1**] J. Milnor, *Dynamics in One Complex Variable: Introductory Lectures*, SUNY at Stony Brook IMS preprint, 1990/5.

[**Mi2**] J. Milnor, *Periodic orbits, external rays, and the Mandelbrot set: An expository account*, see http://www.math.sunysb.edu/˜jack.

[**PM1**] R. Perez-Marco, *Fixed points and circle maps*, Acta Math., **179** (1997) 243-294.
[**PM2**] R. Perez-Marco, *Topology of Julia sets and hedgehogs*, Prepublications Math. Orsay, 1994.
[**Pe**] C. Petersen, *Local connectivity of some Julia sets containing a circle with an irrational rotation*, Acta Math., **177** (1996) 163-224.
[**Po**] C. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer-Verlag, 1992.
[**Ru**] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.
[**S-Z**] D. Schleicher and S. Zakeri, *On biaccessible points in the Julia set of a Cremer quadratic polynomial*, SUNY at Stony Brook IMS preprint, 1998/1(b), to appear in Proc. Amer. Math. Soc.
[**Sø**] D. Sørensen, *Accumulation theorems of quadratic polynomials*, Ergod. Th. and Dynam. Sys., **16** (1996) 555-590.
[**Su**] D. Sullivan, *Conformal Dynamical Systems*, Springer Lecture Notes in Mathematics, **1007** (1983) 725-752.
[**Yo1**] J.C. Yoccoz, *Petits Diviseurs en Dimension 1*, Asterisque **231**, 1995.
[**Yo2**] J.C. Yoccoz, *Recent developments in dynamics*, Proceedings of the International Congress of Mathematicians in Zürich, vol. I, Birkhäuser Verlag, 1994

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